# EE363 homework 1 solutions

1. *LQR for a triple accumulator.* We consider the system $x_{t+1} = Ax_t + Bu_t$, $y_t = Cx_t$, with

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}.$$

This system has transfer function $H(z) = (z-1)^{-3}$, and is called a triple accumulator, since it consists of a cascade of three accumulators. (An accumulator is the discrete-time analog of an integrator: its output is the running sum of its input.) We'll use the LQR cost function

$$J = \sum_{t=0}^{N-1} u_t^2 + \sum_{t=0}^{N} y_t^2,$$

with $N = 50$.

(a) Find $P_t$ (numerically), and verify that the Riccati recursion converges to a steady-state value in fewer than about 10 steps. Find the optimal time-varying state feedback gain $K_t$, and plot its components $(K_t)_{11}$, $(K_t)_{12}$, and $(K_t)_{13}$, versus $t$.

(b) Find the initial condition $x_0$, with norm not exceeding one, that maximizes the optimal value of $J$. Plot the optimal $u$ and resulting $x$ for this initial condition.

*Solution:*

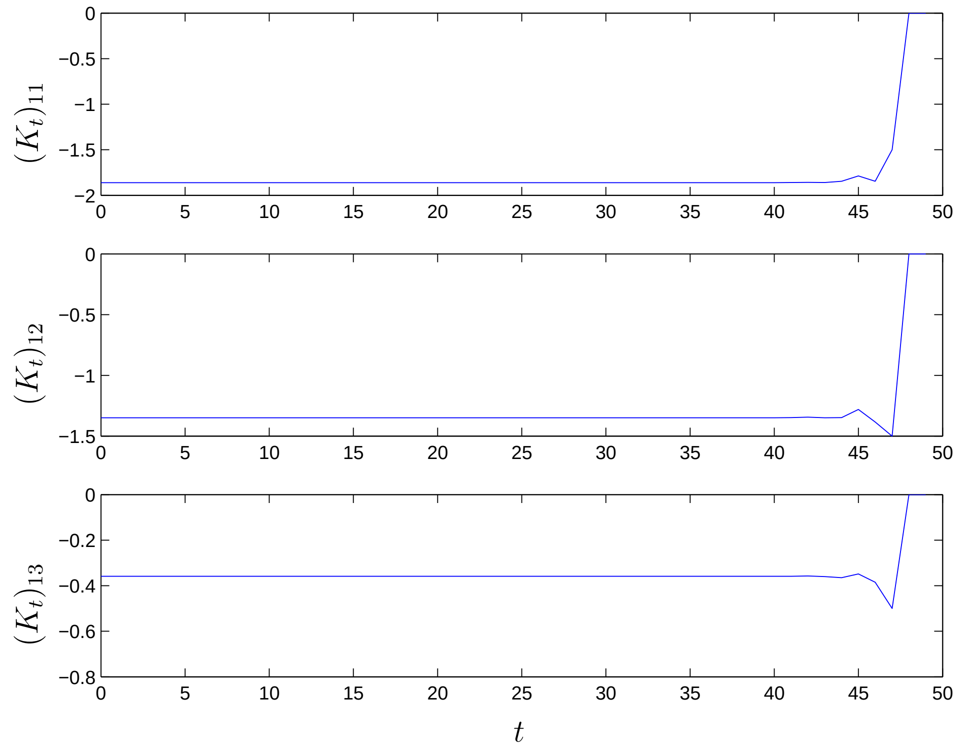
(a) The following Matlab script solves both parts, *i.e.*, implements the Riccati recursion, and finds the worst case initial condition.

```
% LQR for a triple accumulator
% data
A = [1 0 0; 1 1 0; 0 1 1];
B = [1 0 0]';
C = [0 0 1];
Q = C'*C; R = 1;
m=1; n=3; N=50;

% Riccati recursion
P = zeros(n,n,N+1);
P(:,:,N+1) = Q;
K = zeros(m,n,N);
%Nrm = zeros(N,1);
for i = N:-1:1
```

```
    P(:,:,i) = Q+A'*P(:,:,i+1)*A-...
        A'*P(:,:,i+1)*B*pinv(R+B'*P(:,:,i+1)*B)*B'*P(:,:,i+1)*A;
    K(:,:,i) = -pinv(R+B'*P(:,:,i+1)*B)*B'*P(:,:,i+1)*A;
 %   Nrm(N+1-i) = norm(P(:,:,i+1)-P(:,:,i))/norm(P(:,:,i+1));
end

% worst initial condition (max_x(0) min_u J)
[V,D] = eig(P(:,:,1));
x0 = -V(:,1);

% optimal u and resulting x
x = zeros(3,N); x(:,1) = x0;
u = zeros(1,N); u(1) = K(:,:,1)*x(:,1);
for t = 1:N-1
  x(:,t+1) = A*x(:,t) + B*u(t);
  u(t+1) = K(:,:,t+1)*x(:,t+1);
end

% plots
figure(1);
t = 0:49; K = shiftdim(K);
subplot(3,1,1); plot(t,K(1,:)); ylabel('K1(t)');
subplot(3,1,2); plot(t,K(2,:)); ylabel('K2(t)');
subplot(3,1,3); plot(t,K(3,:)); ylabel('K3(t)'); xlabel('t');
%print -depsc tripleaccKt

figure(2);
subplot(4,1,1); plot(t,x(1,:)); ylabel('x1(t)');
subplot(4,1,2); plot(t,x(2,:)); ylabel('x2(t)');
subplot(4,1,3); plot(t,x(3,:)); ylabel('x3(t)');
subplot(4,1,4); plot(t,u); ylabel('u(t)'); xlabel('t');
%print -depsc tripleaccXU

%figure(3);
%semilogy(Nrm); title('Riccati convergence');
```

After about 9 iterations we see that the matrix $P_{t+1} - P_t$ has elements on the order of $10^{-3}$, compared to typical $P_t$ values of around 5.

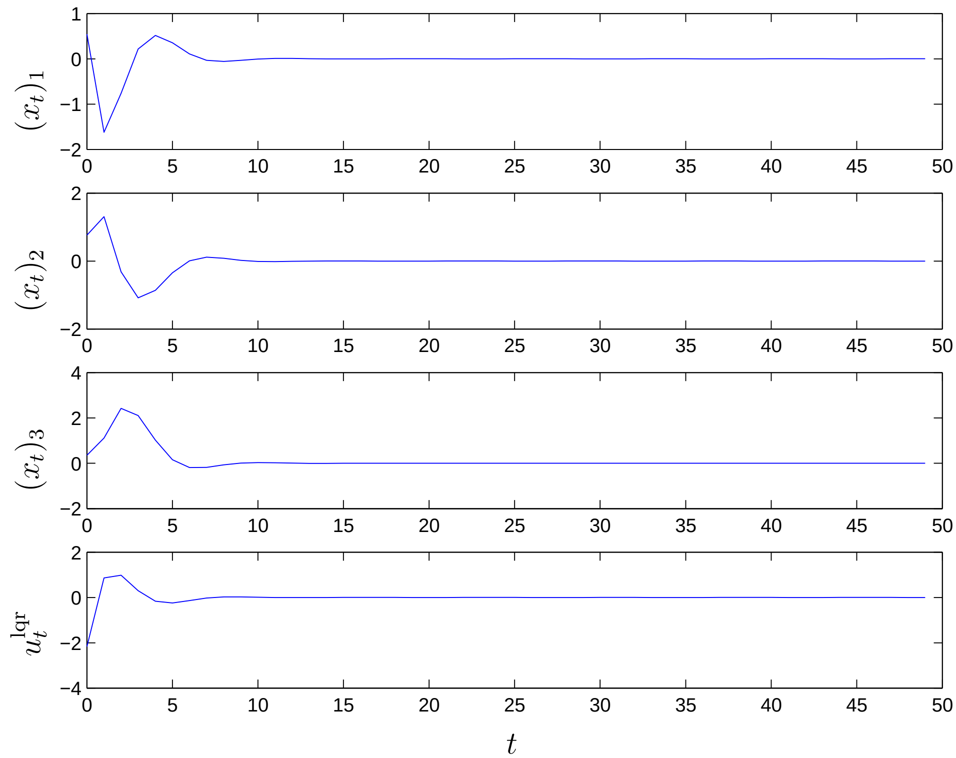
The plot below shows the three elements of $K_t$ versus $t$.




(b) The optimal cost, when started in state $x_0$, is $x_0^T P_0 x_0$. To maximize this we choose $x_0$ as an eigenvector of $P_0$ associated with its maximum eigenvalue. The eigenvector associated with the maximum eigenvalue is

$$x_0^{\max} = \begin{bmatrix} 0.5428 \\ 0.7633 \\ 0.3504 \end{bmatrix}.$$

The optimal trajectories of the three elements of $x_t$ and the optimal input $u_t^{\mathrm{lqr}}$ are shown below.




2. *Linear quadratic state tracking.* We consider the system $x_{t+1} = Ax_t + Bu_t$. In the conventional LQR problem the goal is to make both the state and the input small. In this problem we study a generalization in which we want the state to follow a desired (possibly nonzero) trajectory as closely as possible. To do this we penalize the *deviations* of the state from the desired trajectory, *i.e.*, $x_t - x_t^d$, using the following cost function:

$$J = \sum_{\tau=0}^{N} (x_\tau - x_\tau^d)^T Q (x_\tau - x_\tau^d) + \sum_{\tau=0}^{N-1} u_\tau^T R u_\tau,$$

where we assume $Q = Q^T \geq 0$ and $R = R^T > 0$. (The desired trajectory $x_\tau^d$ is given.) Compared with the standard LQR objective, we have an extra linear term (in $x$) and a constant term.

In this problem you will use dynamic programming to show that the cost-to-go function $V_t(z)$ for this problem has the form

$$z^T P_t z + 2q_t^T z + r_t,$$

with $P_t = P_t^T \geq 0$. (*i.e.*, it has quadratic, linear, and constant terms.)

(a) Show that $V_N(z)$ has the given form.

(b) Assuming $V_{t+1}(z)$ has the given form, show that the optimal input at time $t$ can be written as

$$u_t^\star = K_t x_t + g_t,$$

where

$$K_t = -(R + B^T P_{t+1} B)^{-1} B^T P_{t+1} A, \quad g_t = -(R + B^T P_{t+1} B)^{-1} B^T q_{t+1}.$$

In other words, $u_t^\star$ is an affine (linear plus constant) function of the state $x_t$.

(c) Use backward induction to show that $V_0(z), \ldots, V_N(z)$ all have the given form. Verify that

$$\begin{aligned} P_t &= Q + A^T P_{t+1} A - A^T P_{t+1} B (R + B^T P_{t+1} B)^{-1} B^T P_{t+1} A, \\ q_t &= (A + BK_t)^T q_{t+1} - Q x_t^d, \\ r_t &= r_{t+1} + x_t^d Q x_t^d + q_{t+1}^T B g_t, \end{aligned}$$

for $t = 0, \ldots, N-1$.

*Solution:*

(a) We can write

$$V_N(z) = (z - x_N^d)^T Q (z - x_N^d) = z^T Q z - 2 x_N^d Q z + x_N^d Q x_N^d = z^T P_N z + 2 q_N^T z + r_N,$$

where $P_N = Q$, $q_N = -Q^T x_N^d$ and $r_N = x_N^d Q x_N^d$.

(b) Assuming $V_{t+1}(z)$ has the given form, the Bellman equation can be written as

$$\begin{aligned} V_t(z) &= \min_w \{(z - x_t^d)^T Q (z - x_t^d) + w^T R w + V_{t+1}(Az + Bw)\} \\ &= (z - x_t^d)^T Q (z - x_t^d) + \min_w \{w^T R w \\ &\qquad + (Az + Bw)^T P_{t+1} (Az + Bw) + 2 q_{t+1}^T (Az + Bw) + r_{t+1}\}. \end{aligned}$$

To find the $w$ that minimizes the above expression, we set its derivative with respect to $w$ equal to zero. This gives

$$2Rw + 2B^T P_{t+1}(Az + Bw) + 2B^T q_{t+1} = 0,$$

and so

$$u_t^\star = w^\star = -(R + B^T P_{t+1} B)^{-1} (B^T q_{t+1} + B^T P A z) = K_t z + g_t,$$

where $K_t = -(R + B^T P_{t+1} B)^{-1} B^T P_{t+1} A$, and $g_t = -(R + B^T P_{t+1} B)^{-1} B^T q_{t+1}$.

(c) We can write $V_t(z)$ as

$$\begin{aligned} V_t(z) &= z^T (Q + A^T P_{t+1} A) z + 2 (A^T q_{t+1} - x_t^d)^T z + x_t^d Q x_t^d + r_{t+1} \\ &\qquad + \min_w \{w^T (R + B^T P_{t+1} B) w + 2 (B^T P_{t+1} A z + B^T q_{t+1})^T w\}. \end{aligned}$$

Substituting $w^\star$ into the above expression gives

$$\begin{aligned}
V_t(z) &= z^T(Q + A^T P_{t+1} A)z + 2(A^T q_{t+1} - x_t^d)^T z + x_t^d Q x_t^d + r_{t+1} \\
&\qquad +(B^T q_{t+1} + B^T P_{t+1} A z)^T (R + B^T P_{t+1} B)^{-1} (B^T q_{t+1} + B^T P_{t+1} A z) \\
&\qquad -2(B^T q_{t+1} + B^T P_{t+1} A z)^T (R + B^T P_{t+1} B)^{-1} (B^T q_{t+1} + B^T P_{t+1} A z) \\
&= z^T(Q + A^T P_{t+1} A)z + 2(A^T q_{t+1} - x_t^d)^T z + x_t^d Q x_t^d + r_{t+1} \\
&\qquad -(B^T q_{t+1} + B^T P_{t+1} A z)^T (R + B^T P_{t+1} B)^{-1} (B^T q_{t+1} + B^T P_{t+1} A z).
\end{aligned}$$

After rearranging and collecting terms we get

$$\begin{aligned}
V_t(z) &= z^T(Q + A^T P_{t+1} A - A^T P_{t+1} B (R + B^T P_{t+1} B)^{-1} B^T P_{t+1} A) z \\
&\qquad +2(A^T q_{t+1} - A^T P_{t+1} B (R + B^T P_{t+1} B)^{-1} B^T q_{t+1} - x_t^d)^T z \\
&\qquad\quad +r_{t+1} + x_t^d Q x_t^d - q_{t+1}^T B (R + B^T P_{t+1} B)^{-1} B^T q_{t+1} \\
&= z^T P_t z + 2 q_t^T z + r_t,
\end{aligned}$$

where

$$\begin{aligned}
P_t &= Q + A^T P_{t+1} A - A^T P_{t+1} B (R + B^T P_{t+1} B)^{-1} B^T P_{t+1} A, \\
q_t &= A^T q_{t+1} - A^T P_{t+1} B (R + B^T P_{t+1} B)^{-1} B^T q_{t+1} - x_t^d = (A + BK)^T q_{t+1} - x_t^d, \\
r_t &= r_{t+1} + x_t^d Q x_t^d - q_{t+1}^T B (R + B^T P_{t+1} B)^{-1} B^T q_{t+1} = r_{t+1} + x_t^d Q x_t^d + q_{t+1}^T B g_t.
\end{aligned}$$

Thus we have shown that if $V_{t+1}(z)$ has the given form, then $V_t(z)$ also has the given form. Since $V_N(z)$ has this form (from part (a)), by induction, $V_0(z), \ldots, V_N(z)$ all have the given form.

3. *The Schur complement.* In this problem you will show that if we minimize a positive semidefinite quadratic form over *some* of its variables, the result is a positive semidefinite quadratic form in the *remaining* variables. Specifically, let

$$J(u,z) = \begin{bmatrix} u \\ z \end{bmatrix}^T \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} \begin{bmatrix} u \\ z \end{bmatrix}$$

be a positive semidefinite quadratic form in $u$ and $z$. You may assume $Q_{11} > 0$ and $Q_{11}, Q_{22}$ are symmetric. Define $V(z) = \min_u J(u,z)$. Show that $V(z) = z^T P z$, where $P$ is symmetric positive semidefinite (find $P$ explicitly).

The matrix $P$ is called the *Schur complement* of the matrix $Q_{11}$ in the big matrix above. It comes up in many contexts.

*Solution:* We can write

$$J(u,z) = u^T Q_{11} u + 2u^T Q_{12} z + z^T Q_{22} z.$$

To find the $u$ that minimizes this expression, we set its derivative with respect to $u$ equal to zero. This gives

$$2Q_{11}u + 2Q_{12}z = 0$$

and so $u^\star = -Q_{11}^{-1}Q_{12}z$. Thus we get

$$V(z) = z^TQ_{12}^TQ_{11}^{-1}Q_{12}z + z^TQ_{22}z - 2z^TQ_{12}^TQ_{11}Q_{12}z = z^T(Q_{22} - Q_{12}^TQ_{11}^{-1}Q_{12})z = z^TPz,$$

where

$$P = Q_{22} - Q_{12}^TQ_{11}^{-1}Q_{12}.$$

Clearly, $P$ is symmetric; it is also positive semidefinite, since

$$z^TPz = \min_u J(u,z) \geq 0, \quad \forall z.$$

4. *A useful determinant identity.* Suppose $X \in \mathbf{R}^{n\times m}$ and $Y \in \mathbf{R}^{m\times n}$.

   (a) Show that $\det(I + XY) = \det(I + YX)$. *Hint:* Find a block lower triangular matrix $L$ for which

   $$\begin{bmatrix} I & X \\ -Y & I \end{bmatrix} = L \begin{bmatrix} I & X \\ 0 & I \end{bmatrix},$$

   and use this factorization to evaluate the determinant of this matrix. Then find a block upper triangular matrix $U$ for which

   $$\begin{bmatrix} I & X \\ -Y & I \end{bmatrix} = U \begin{bmatrix} I & 0 \\ -Y & I \end{bmatrix},$$

   and repeat.

   (b) Show that the nonzero eigenvalues of $XY$ and $YX$ are exactly the same.

*Solution:*

(a) Multiplying on the right by two different matrices yields

$$\begin{aligned} \det \begin{bmatrix} I & X \\ -Y & I \end{bmatrix} &= \det\left( \begin{bmatrix} I & 0 \\ -Y & I+YX \end{bmatrix} \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \right) &= \det(I + YX) \\ &= \det\left( \begin{bmatrix} I+XY & X \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ -Y & I \end{bmatrix} \right) &= \det(I+XY). \end{aligned}$$

(b) Recall that an eigenvalue $\lambda$ of a matrix $A$ satisfies $\det(\lambda I - A) = 0$. Let $I_r$ denote an $r \times r$ identity matrix. If $\lambda$ is a nonzero eigenvalue of $XY$,

$$\begin{aligned} 0 &= \det(\lambda I_n - XY) \\ &= \det\left(\lambda I_n(I_n + (-\frac{1}{\lambda}X)Y)\right) \\ &= \det(\lambda I_n)\det(I_n + (-\frac{1}{\lambda}X)Y). \end{aligned}$$

We know

$$\det(\lambda I_n) = \lambda^n = \lambda^{n-m}\det(\lambda I_m),$$

and from part (a)

$$\det(I_m + Y(-\frac{1}{\lambda}X)) = \det(I_n + (-\frac{1}{\lambda}X)Y),$$

so

$$\begin{aligned} 0 &= \lambda^{n-m}\det(\lambda I_m)\det(I_m + (-\frac{1}{\lambda})YX) \\ &= \lambda^{n-m}\det(\lambda I_m - YX). \end{aligned}$$

Since $\lambda$ is nonzero, we have that $\det(\lambda I_m - YX) = 0$. Hence $\lambda$ is also an eigenvalue of $YX$. By similar argument, if $\lambda$ is a nonzero eigenvalue of $YX$, we get that $0 = \det(\lambda I_n - XY)$, showing $\lambda$ is also an eigenvalue of $XY$. Thus, $XY$ and $YX$ have exactly the same nonzero eigenvalues.

5. *When does a finite-horizon LQR problem have a time-invariant optimal state feedback gain?* Consider a discrete-time LQR problem with horizon $t = N$, with optimal input $u(t) = K_t x(t)$. Is there a choice of $Q_f$ (symmetric and positive semidefinite, of course) for which $K_t$ is constant, *i.e.*, $K_0 = \cdots = K_{N-1}$?

   *Solution:* $K_t$ is defined by $K_t = -(R + B^T P_{t+1} B)^{-1} B^T P_{t+1} A$. Therefore, if $P_1 = \cdots = P_N$, we have $K_0 = \cdots = K_{N-1}$. Since $Q_f = P_N$, and for $t$ far from $N$ we have that $P_t$ converges to the steady-state solution, we see that to have constant $P_t$, we must have $Q_f$ equal to the steady-state solution $P_{ss}$. With this choice, we have $P_t = P_{ss}$ for all $t$, and therefore $K_t = K_{ss}$ for all $t$.