EE363 homework 3 solutions

1. *Solution of a two-point boundary value problem.* We consider a linear dynamical system $\dot{x} = Ax$, with $x(t) \in \mathbf{R}^n$. There is an n-dimensional subspace of solutions of this equation, so to single out one of the trajectories we can impose, roughly speaking, n equations.

   In the most common situation, we specify $x(0) = x_0$, in which case the unique solution is $x(t) = e^{tA}x_0$. This is called an *initial value problem* since we specify the initial value of the state. In a *final value problem*, we specify the final state: $x(T) = x_f$. In this case the unique solution is $x(t) = e^{(t-T)A}x_f$.

   In a *two-point boundary value problem* we impose conditions on the initial and final states.

   (a) Find the solution to the two-point boundary value problem

   $$\dot{x} = Ax, \qquad Fx(0) + Gx(T) = h.$$

   Here $F,\ G \in \mathbf{R}^{n\times n}$, and $h \in \mathbf{R}^n$. Express your answer in terms of A, F, G, and h. Your answer can contain a matrix exponential.

   What condition must hold to ensure that there is a unique solution to this equation?

   (b) Express the two-point boundary value problem that arises in the continuous time LQR problem (*i.e.*, with the Hamiltonian system) in the form given above, and then find the solution to this boundary value problem. (You may leave matrix exponentials in your solution.) How is the optimal input u obtained from this solution?

   Solution:

   (a) $x(T)$ can be expressed as $x(T) = e^{AT}x(0)$. The boundary conditions are then

   $$Fx(0) + Gx(T) = (F + Ge^{AT})x(0) = h$$

   If $(F+Ge^{AT})$ is nonsingular, the unique solution to the two-point boundary value problem is

   $$x(t) = e^{At}x(0) = e^{At}(F + Ge^{AT})^{-1}h$$

   (b) The Hamiltonian system is given by $\dot{v} = Hv$, where

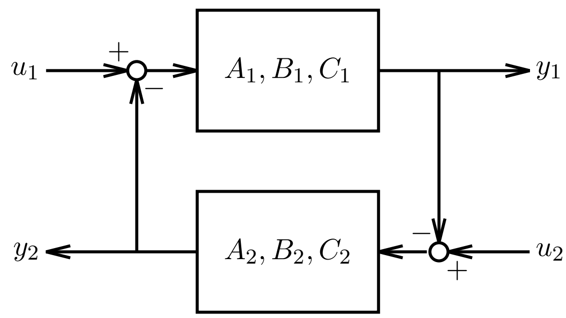
   $$v(t) = \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} \qquad H = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix}$$

The boundary conditions for this problem are $x(0) = x_0$ and $\lambda(T) = Q_f x(T)$. These can be expressed as $Fv(0) + Gv(T) = h$, where

$$F = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \qquad G = \begin{bmatrix} 0 & 0 \\ -Q_f & I \end{bmatrix} \qquad h = \begin{bmatrix} x_0 \\ 0 \end{bmatrix}$$

The solution to this boundary value problem is $v(t) = e^{Ht}(F + Ge^{HT})^{-1}h$. Since $u(t) = -R^{-1}B^T\lambda(t)$, the optimal input is $u(t) = -R^{-1}B^T[\ 0 \quad I\ ]e^{Ht}(F+Ge^{HT})^{-1}h$.

2. *Controllability of a feedback connection.* Consider the feedback connection of two linear dynamical systems:



(a) Write state (linear dynamical system) equations with state $x = (x_1, x_2)$, input $u = (u_1, u_2)$, and output $y = (y_1, y_2)$.

(b) *Proof or counterexample:* the feedback system is observable (controllable) if and only if both subsystems are observable (controllable).

(c) Fix $u_2 = 0$ (*i.e.*, u_2 is not an input), and repeat part (b).

Solution:

(a) By inspection,

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A_1 & -B_1C_2 \\ -B_2C_1 & A_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

(b) This fact is true. To see this, we factor the PBH observability matrix:

$$\begin{bmatrix} C_1 & 0 \\ 0 & C_2 \\ sI - A_1 & B_1C_2 \\ B_2C_1 & sI - A_2 \end{bmatrix} = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & B_1 & I & 0 \\ B_2 & 0 & 0 & I \end{bmatrix}\begin{bmatrix} C_1 & 0 \\ 0 & C_2 \\ sI - A_1 & 0 \\ 0 & sI - A_2 \end{bmatrix}$$

The left-hand factor is full-rank, and can be ignored; the right-hand factor is precisely the PBH matrix for the diagonal connection of two systems. It is obvious that this matrix is full-rank for all s if and only if (C_1, A_1) and (C_2, A_2) are both observable.

The proof for controllability follows the same method.

(c) Fixing $u_2 = 0$ is equivalent to eliminating the B_2 column of the B matrix of the feedback connection (without altering the A matrix). This obviously has no effect on the observability of the system.

The controllability of the feedback connection, however, *is* affected; and now part (b) is *false*. The controllability of both systems is still a necessary condition, but it is no longer sufficient. An obvious counterexample is

$$A_1 = A_2 = B_1 = B_2 = C_2 = 1, \quad C_1 = 0$$
$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

In this case, (A_1, B_1) and (A_2, B_2) are controllable, but (A, B) is not.

3. You know that $\mathcal{R}(\mathcal{C})$ is A-invariant, where $\mathcal{C} = [B \ AB \ \cdots \ A^{n-1}B]$ is the controllability matrix. Find a matrix X such that $A\mathcal{C} = \mathcal{C}X$.

Solution: We have

$$A\mathcal{C} = A\begin{bmatrix} B & AB & \ldots & A^{n-1}B \end{bmatrix} = \begin{bmatrix} AB & A^2B & \ldots & A^nB \end{bmatrix}$$

We need to express this matrix as $\mathcal{C}X$, for some X:

$$\mathcal{C}X = \begin{bmatrix} B & AB & \ldots & A^{n-1}B \end{bmatrix} \begin{bmatrix} 0 & 0 & \ldots & K_0 \\ I & 0 & \ldots & K_1 \\ 0 & I & \ldots & K_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & K_{n-1} \end{bmatrix}.$$

That leaves the last column of $\mathcal{C}X$ as

$$K_0 I + ABK_1 + \ldots + A^{n-1}BK_{n-1}.$$

We would like to choose the K's such that this equals A^nB. By the Cayley-Hamilton theorem we know that

$$A^n = -\alpha_0 I - \alpha_1 A - \ldots - \alpha_{n-1}A^{n-1},$$

where α_i is the ith coefficient of A's characteristic polynomial, multiplying on the right with B, we get

$$A^nB = -\alpha_0 B - \alpha_1 AB - \ldots - \alpha_{n-1}A^{n-1}B,$$

giving the K_i matricies as

$$K_i = -\alpha_i I.$$

So our X matrix is

$$\begin{bmatrix} 0 & 0 & \dots & -\alpha_0 I \\ I & 0 & \dots & -\alpha_1 I \\ 0 & I & \dots & -\alpha_2 I \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -\alpha_{n-1} I \end{bmatrix}.$$

4. *A matrix criterion for A-invariance of a nullspace.* We saw in lecture that $\mathcal{R}(M)$ is A-invariant if and only if there exists a matrix X such that $AM = MX$. In this problem you will derive a similiar condition for the nullspace of a matrix.

   (a) Show that $\mathcal{N}(N)$ is A-invariant if and only if $\mathcal{R}(N^T)$ is A^T-invariant.

   (b) Finish the statement "$\mathcal{N}(N)$ is A-invariant if and only if ...".

   Solution:

   (a) Recall that $\mathcal{R}(N^T)$ and $\mathcal{N}(N)$ are complementary subspaces. Suppose $\mathcal{V}$ is A-invariant and is spanned by the columns of the $\mathcal{N}(N)$. Since $(\mathcal{V}^\perp)^\perp = \mathcal{V}$, $(A^T)^T = A$, we only need to show one direction.
   For every $v \in \mathcal{V}, z \in \mathcal{V}^\perp$,

   $$(A^T z)^T v = z^T \underbrace{Av}_{\in \mathcal{V}} = 0,$$

   i.e., $A^T z \in \mathcal{V}^\perp$. Hence $\mathcal{V}^\perp$ is A^T-invariant.

   (b) Because $\mathcal{N}(N)^\perp = \mathcal{R}(N^T)$, $\mathcal{N}(N)$ is A-invariant if and only if $\mathcal{R}(N^T)$ is A^T-invariant, *i.e.*, there exists a matrix X such that $A^T N^T = N^T X^T \iff NA = XN$. And so the correct statement is: $\mathcal{N}(N)$ is A-invariant if and only if there exists a matrix X such that $NA = XN$.

5. *Complex eigenvalues and invariant planes.* Let $A \in \mathbf{R}^{n \times n}$ satisfy $Av = (\alpha + j\beta)v$, where $v \in \mathbf{C}^n$ is nonzero, and $\beta \neq 0$. (In other words, v is an eigenvector of A corresponding to an eigenvalue that is not real.)

   (a) Show that $\text{span}\{\Re v, \Im v\}$ is an A-invariant subspace.

   (b) With $M = [\Re v \ \Im v]$, find $X \in \mathbf{R}^{2 \times 2}$ such that $AM = MX$.

   Solution:

(a)

$$\begin{aligned} Av &= A(\Re v + j\Im v) = (\alpha + j\beta)(\Re v + j\Im v) \iff \\ A\Re v &= \alpha\Re v - \beta\Im v \\ A\Im v &= \beta\Re v + \alpha\Im v \end{aligned}$$

Since any element of $\mathbf{Span}\{\Re v, \Im v\}$ can be represented by $\hat{v} = a_1\Re v + a_2\Im v$,

$$\begin{aligned} A\hat{v} &= a_1 A\Re v + a_2 A\Im v \\ &= a_1(\alpha\Re v - \beta\Im v) + a_2(\beta\Re v + \alpha\Im v) \in \mathbf{Span}\{\Re v, \Im v\} \end{aligned}$$

(b) Using the expressions for $A\Re v$ and $A\Im v$ above,

$$\begin{aligned} A[\Re v\ \Im v] &= [(\alpha\Re v - \beta\Im v)\ (\beta\Re v + \alpha\Im v)] \\ &= [\Re v\ \Im v]\begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \end{aligned}$$

6. *Stochastic LQR for a supply chain system with manufacturing delay.* We let $s_t \in \mathbf{R}$, $t = 1, 2, \ldots$, denote the stock level of some product available at the beginning of period t. This is measured with respect to some nominal value, so negative values just mean our stock is below the nominal value. We let o_t, $t = 1, 2, \ldots$, denote the amount of product ordered (say, from a factory) at the beginning of period t. The demand for the product in period t will be denoted d_t. The amount ordered and demand are also relative to some nominal values, so, *e.g.*, negative d_t just means that in period t there is less than the nominal demand for the product. We assume that the demands d_t are IID with zero mean and variance σ^2.

   When the order o_t is placed, you can assume that the current and previous stock levels $s_t, s_{t-1}, \ldots$ are known, as are the previous orders $o_{t-1}, o_{t-2}, \ldots$, and the previous demand levels $d_{t-1}, d_{t-2}, \ldots$. But the current period demand d_t is not known when the order o_t is placed.

   The stock level propagates as

$$s_{t+1} = s_t + o_{t-D+1} - d_t,$$

   where D (a nonnegative integer) gives the delay between placing an order and receiving the product. For example, with $D = 3$, product ordered in period t arrives in stock in period $t + D$. In the equation above, we interpret o_t as zero for $t \le 0$.

   The objective J, which is to be minimized, is

$$J = \lim_{T\to\infty} \frac{1}{T} \mathbf{E}\left(\sum_{t=1}^{T} s_t^2 + \rho o_t^2\right),$$

where $\rho > 0$. We can interpret J as the mean-square deviation of the stock level from the nominal value, plus ρ times the mean-square deviation of the orders from the nominal value.

(a) Explain how to find an optimal ordering policy. We do not need you to find an analytical solution: Your method can be computational, *e.g.*, involve solution of an ARE. Be sure to say what form the optimal ordering policy has. (For example, is it a linear function of s_t?)

Hint. Define a state that allows you to formulate the problem as a standard stochastic LQR problem.

(b) Find an optimal ordering policy for $\sigma = 1$, $\rho = 1$ and $D = 3$, and give the optimal mean-square deviation $J^\star$.

(c) Simulate the closed-loop system, using the optimal ordering policy, for 15000 steps, then discard the first 5000 steps, to get rid of the initial transient effect. Plot a histogram of the stage cost $s_t^2 + \rho o_t^2$, and find the (empirical) mean-square value, comparing to the (exact) value of $J^\star$. Plot a trace of s_t, o_t, and d_t over (say) 50 steps.

Solution.

(a) Let's define a state $x_t = (s_t, o_{t-D+1}, o_{t-D+2}, \ldots, o_{t-1}) \in \mathbf{R}^D$. We can write

$$
\begin{aligned}
x_{t+1} = \begin{bmatrix} s_{t+1} \\ o_{t-D+2} \\ o_{t-D+3} \\ \vdots \\ o_t \end{bmatrix} &= \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} s_t \\ o_{t-D+1} \\ o_{t-D+2} \\ \vdots \\ o_{t-1} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} o_t + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} d_t \\
&= Ax_t + Bo_t + w_t,
\end{aligned}
$$

where

$$
A = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad w_t = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} d_t.
$$

The random variables w_t are IID with zero mean and covariance $W = \mathbf{diag}(\sigma^2 e_1)$. The objective becomes

$$
J = \lim_{T\to\infty} \frac{1}{T} \mathbf{E} \left(\sum_{t=1}^{T} x_t^T Q x_t + \rho o_t^2 \right),
$$

where $Q = \mathbf{diag}(e_1)$.

So we see that by defining a state x_t, that consists of the current stock s_t, and the input history $o_{t-D+1}, \ldots, o_{t-1}$, we have written the problem as a standard infinite horizon linear stochastic control problem (see lecture 5). Thus the optimal cost is

$$J^\star = \mathbf{Tr}(WP),$$

where P satisfies the ARE

$$P = Q + A^T PA - A^T PB(\rho + B^T PB)^{-1} B^T PA.$$

An optimal policy is

$$o_t = -(\rho + B^T PB)^{-1} B^T PAx_t = Kx_t.$$

In terms of the original variables, we can write this as

$$o_t = K_{11}s_t + K_{12}o_{t-D+1} + \cdots + K_{1D}o_{t-1},$$

i.e., a linear function of the current stock level, plus a linear function of the previous D orders.

(b) The following matlab code solves the problem (for part (b) and part (c)).

```
randn('state',0);
% problem data
D = 3;
sigma = 1;
rho = 1;

% define linear dynamical system
e1 = [1;zeros(D-1,1)];
A = [[1;zeros(D-2,1)],eye(D-1);0,zeros(1,D-1)];
B = [zeros(D-1,1);1];
W = diag(sigma^2*e1);
Q = diag(e1);

% solve the Riccati equation by fixed point iteration
P = eye(D);
for i = 1:100
    P = Q+A'*P*A-A'*P*B*pinv(rho+B'*P*B)*B'*P*A;
end

K = -pinv(rho+B'*P*B)*B'*P*A;
Jstar = trace(W*P);
```

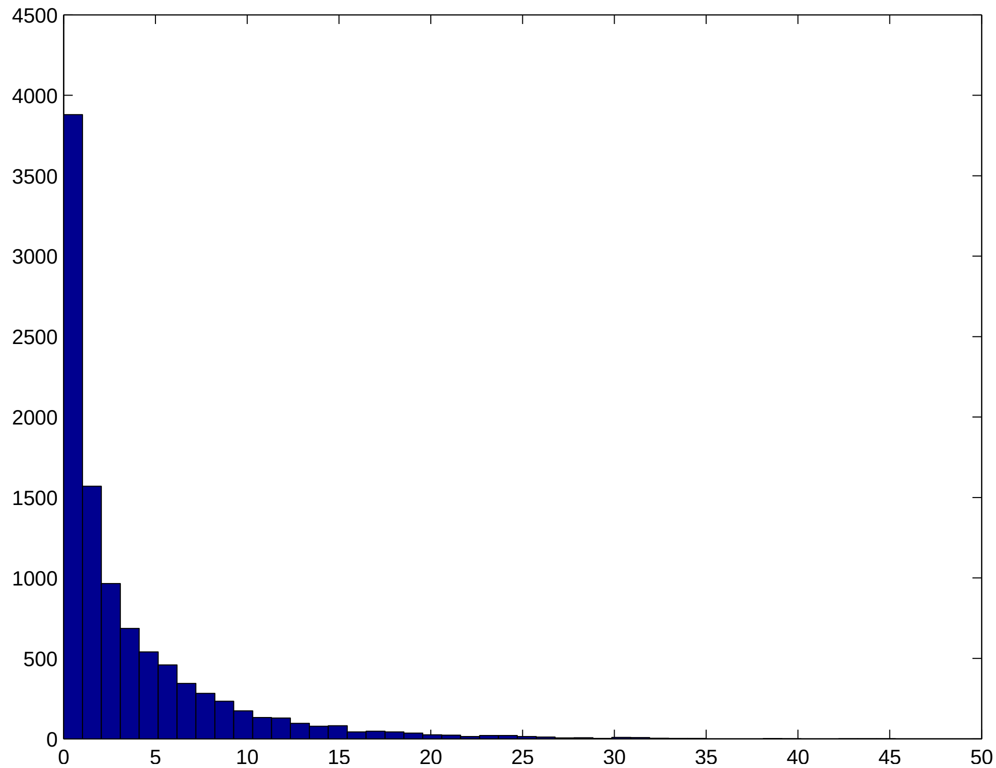
```
nsim = 15000;    % number of simulation steps
x = zeros(D,1); % initial state
J = zeros(nsim,1); s = zeros(nsim,1);
o = zeros(nsim,1); d = zeros(nsim,1);
Whalf = sqrtm(W);
for i = 1:nsim
    s(i) = x(1); o(i) = K*x; d(i) = sigma*randn;
    J(i) = x(1)^2+rho*o(i)^2;
    x = A*x+B*o(i)+e1*d(i);
end

J(1:5000) = []; % discard the first 5000 steps
Jemp = mean(J); % compute the empirical mean

% plot histogram
figure; hist(J,50);
axis([0,50,0,4500]);
print('-depsc','supply_stoch_hist.eps');
tvec = [0:1:50]';

% plot sample trace
figure;
subplot(3,1,1); stairs(tvec,s(1:51)); ylabel('s');
subplot(3,1,2); stairs(tvec,o(1:51)); ylabel('o');
subplot(3,1,3); stairs(tvec,d(1:51)); ylabel('d');
xlabel('t');
print('-depsc','supply_stoch_trace.eps');
```

For $\sigma = 1$, $\rho = 1$, and $D = 3$, the optimal policy is

$$o_t = -0.6180 s_t - 0.6180 o_{t-2} - 0.6180 o_{t-1},$$

and the optimal cost is $J^\star = 3.6180$.

(c) The following figure shows the histogram of stage costs for a 10000 step simulation. The empirical average of the stage costs (over the 10000) steps is 3.6087.



Here is a sample trace of s_t, o_t, and d_t over 50 steps.