EE363 homework 7 solutions

1. *Gain margin for a linear quadratic regulator.* Let K be the optimal state feedback gain for the LQR problem with system $\dot{x} = Ax + Bu$, state cost matrix $Q \geq 0$, and input cost matrix $R > 0$. You can assume that (A, B) is controllable and (Q, A) is observable.

   We consider the system
   $$\dot{x} = Ax + Bu, \qquad u = \alpha K x,$$
   where $\alpha > 0$. If $\alpha = 1$, this gives the LQR optimal input, but for $\alpha \neq 1$ this input is obviously not LQR optimal, and indeed the closed-loop system $\dot{x} = (A + \alpha BK)x$ can even be unstable.

   Show that for $\alpha > 1/2$, the closed-loop system is stable. In classical control theory, the ability of a system to remain stable when the input is scaled by any factor in the interval $(1/2, \infty)$ is described as a negative gain margin of 6dB, and an infinite positive gain margin.

   Hint. Use the obvious quadratic Lyapunov function.

   Solution: Let P be the positive definite solution of the ARE $A^T P + PA = K^T RK - Q$. We will use $V(z) = z^T Pz$ as our Lyapunov function. With $\dot{x} = (A + \alpha BK)x$, we have $\dot{V}(z)$ is quadratic, with quadratic form
   $$\begin{aligned}(A + \alpha BK)^T P + P(A + \alpha BK) &= A^T P + PA - 2\alpha K^T RK \\ &= -(Q + (2\alpha - 1)K^T RK).\end{aligned}$$
   Since $Q \geq 0$ and $K^T RK \geq 0$ for $\alpha \geq 1/2$ we have $(Q + (2\alpha - 1)K^T RK) \geq 0$. To conclude stability, we now just need to show that $(Q + (2\alpha - 1)K^T RK, A + \alpha BK)$ is observable, *i.e.*, there is no $v \neq 0$ for which $(A + \alpha BK)v = \lambda v$ and $(Q + (2\alpha - 1)K^T RK)v = 0$. If there were such a v, we would have
   $$v^T (Q + (2\alpha - 1)K^T RK)v = v^T Qv + (2\alpha - 1)v^T K^T RKv = 0,$$
   which implies $Qv = 0$ and $Kv = 0$. This would give
   $$(A + BK)v = Av = \lambda v, \qquad Qv = 0,$$
   which is not possible since we've assumed (Q, A) observable.

2. *Gradient systems.* Suppose ϕ is a scalar valued function on $\mathbf{R}^n$. (You can assume it is smooth, or has any other technical property you need.) We can define several dynamical systems using the gradient of ϕ. These systems are sometimes called *gradient systems*, and in this context, ϕ is sometimes called the *potential function*.

(a) A second order gradient system has the form $\ddot{x} = -\nabla\phi(x)$. Show that $V(x(t)) = \phi(x(t)) + (1/2)\|\dot{x}(t)\|_2^2$ is a conserved quantity.

(b) A first order gradient system has the form $\dot{x} = -\nabla\phi(x)$. Show that ϕ is a dissipated quantity.

(c) Suppose ϕ has bounded sublevel sets. Show that for every solution of $\dot{x} = -\nabla\phi(x)$, we have $\nabla\phi(x(t)) \to 0$ as $t \to \infty$.
Hint. Show that $\int_0^\infty \|\nabla\phi(x(t))\|^2 \, dt < \infty$.

Solution:

(a) To check that $V(x(t))$ is constant along any trajectory, we compute its time derivative:

$$\dot{V}(x(t)) = \frac{d}{dt}\left(\phi(x(t)) + \frac{1}{2}\dot{x}(t)^T\dot{x}(t)\right) = \nabla\phi(x(t))^T\dot{x}(t) + \ddot{x}(t)^T\dot{x}(t) = 0.$$

Therefore, trajectories of the second order gradient system stay on level surfaces of V, $\{z \in \mathbf{R}^n \mid V(z) = a\}$.

(b) We claim that function $V(x(t)) = \phi(x(t))$ is a dissipated quantity, since

$$\dot{V}(x(t)) = \frac{d}{dt}(\phi(x(t))) = \nabla\phi(x(t))^T\dot{x}(t) = -\|\nabla\phi(x(t))\|^2 \leq 0.$$

Therefore, trajectories of the first order gradient system can only stay the same as current value or slip down to lower values of V. All sublevel sets $\{z \in \mathbf{R}^n \mid V(z) \leq a\}$ are invariant.

(c) Following the hint, we will show that

$$\int_0^\infty \|\nabla\phi(x(t))\|^2 \, dt < \infty,$$

then, assuming $\nabla\phi$ is continuous, we must have $\nabla\phi(x(t)) \to 0$ as $t \to \infty$. From part (b), $\dot{\phi}(x(t)) = -\|\nabla\phi(x(t))\|^2$, so we can write

$$\int_0^t \|\nabla\phi(x(\tau))\|^2 \, d\tau = \int_0^t -\dot{\phi}(x(\tau)) \, d\tau = \phi(x(0)) - \phi(x(t)).$$

Since $\dot{\phi}(x(t)) \leq 0$, for any t we have $x(t) \in \mathcal{C}_0$, where $\mathcal{C}_0$ is the $\phi(x(0))$-sublevel set of ϕ, *i.e.*,

$$\mathcal{C}_0 = \{\ z \mid \phi(z) \leq \phi(x(0))\ \}.$$

$\mathcal{C}_0$ is bounded, which means we can find a compact set $\mathcal{C}$, such that $\mathcal{C}_0 \subseteq \mathcal{C}$. So we get

$$\begin{aligned}
\int_0^t \|\nabla\phi(x(t))\|^2 \, dt &\leq \sup_t \left(\phi(x(0)) - \phi(x(t))\right) \\
&= \phi(x(0)) - \inf_t \phi(x(t)) \\
&\leq \phi(x(0)) - \inf_{z\in\mathcal{C}} \phi(z) \\
&= B < \infty.
\end{aligned}$$

The last inequality follows from the fact that the infimum of a real valued function over a compact set is finite. Since the integral is uniformly bounded by B for all t, we conclude that

$$\int_0^\infty \|\nabla\phi(x(t))\|^2 \, dt \le B < \infty.$$

3. *A bound on peaking factor via Lyapunov theory.* Consider the system $\dot{x} = f(x)$, where $f : \mathbf{R}^n \to \mathbf{R}^n$, and $f(0) = 0$ (*i.e.*, 0 is an equilibrium point).

We define the *peaking factor* p of the system as

$$p = \sup_{t \ge 0,\ x(0) \ne 0} \frac{\|x(t)\|}{\|x(0)\|},$$

where the supremum is taken over all trajectories of the system with $x(0) \ne 0$. We can have $p = \infty$, which would occur, for example, if there is an unbounded trajectory of the system. We always have $p \ge 1$; $p = 1$ only if all trajectories never increase in norm. The peaking factor gives a bound on how far any trajectory can wander away from the origin, relative to how far it was when it started.

In this problem you will show how to bound the peaking factor of a system using Lyapunov methods.

Suppose the quadratic Lyapunov function $V(z) = z^T P z$, $P = P^T > 0$, satisfies $\dot{V}(z) = 2z^T P f(z) \le 0$ for all z. (In other words, the Lyapunov function V proves that all trajectories are bounded.) Show that $p \le \sqrt{\kappa}$, where $\kappa = \lambda_{\max}(P)/\lambda_{\min}(P)$ is the condition number of P.

Two interpretations/ramifications of this result:

- By finding a quadratic Lyapunov function that proves the trajectories are bounded, we can bound the peaking factor of the system.
- If a system has a large peaking factor, then any quadratic Lyapunov function that proves boundedness of the trajectories must have large condition number.

Solution. Since $\dot{V}(z) \le 0$ for all z, $V(x(t))$ is nonincreasing; in particular, $V(x(t)) \le V(x(0))$ for $t \ge 0$. We combine this with the basic inequalities

$$\begin{aligned} V(x(t)) &= x(t)^T P x(t) \ge \lambda_{\min}(P)\|x(t)\|^2, \\ V(x(0)) &= x(0)^T P x(0) \le \lambda_{\max}(P)\|x(0)\|^2 \end{aligned}$$

to get

$$\lambda_{\min}(P)\|x(t)\|^2 \le \lambda_{\max}(P)\|x(0)\|^2$$

for all $t \ge 0$. Assuming $x(0) \ne 0$, this yields

$$\frac{\|x(t)\|}{\|x(0)\|} \le \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} = \sqrt{\kappa}.$$

Therefore, we have $p \le \sqrt{\kappa}$.

4. *Digital filter with saturation.* The dynamics of an undriven digital filter which exhibits saturation is $x(k+1) = \mathbf{sat}(Ax(k))$, where $A \in \mathbf{R}^{n\times n}$ is stable, and $\mathbf{sat} : \mathbf{R}^n \to \mathbf{R}^n$ is defined by $\mathbf{sat}(x) = (\mathbf{sat}(x_1), \ldots, \mathbf{sat}(x_n))$, and the (unit) *saturation function* is defined by

$$\mathbf{sat}(z) = \begin{cases} z & |z| < 1 \\ 1 & z \geq 1 \\ -1 & z \leq -1. \end{cases}$$

Without saturation, the state would converge to zero, but with saturation, it need not. When a trajectory of the system fails to converge to zero, it is called *saturation induced instability.* We seek conditions under which the system with saturation is globally asymptotically stable, *i.e.*, saturation induced instability cannot occur.

Show that $x(k) \to 0$ if there is a nonsingular diagonal D such that $\|DAD^{-1}\| < 1$. (We will see later how to compute such a D, or determine that none exists, using linear matrix inequalities.)

Hint: Use the Lyapunov function $V(z) = \|Dz\|$.

Solution: This condition can be obtained by finding an appropriate Lyapunov function for this system, *i.e.*, a positive definite function $V(z)$ such that $\Delta V = V(\mathbf{sat}(Az)) - V(z) \leq -\alpha V(z)$ for some $\alpha > 0$.

First note that in the scalar case, $|\mathbf{sat}(z)| \leq |z|$. For example, when z doesn't saturate, then equality is achieved. The $\mathbf{sat}(\cdot)$ function just extends this idea to vectors in $\mathbf{R}^n$:

$$\|\mathbf{sat}(z)\|^2 = \sum_{i=1}^{n} (\mathbf{sat}(z_i))^2 \leq \sum_{i=1}^{n} (z_i)^2 = \|z\|^2.$$

In fact, $\|D(\mathbf{sat}(z))\| \leq \|Dz\|$ for any diagonal matrix D.

Let $V(z) = \|Dz\|$, where D is invertible. This gives

$$\begin{aligned} V(\mathbf{sat}(Az)) &= \|D\mathbf{sat}(Az)\| \\ &\leq \|DAz\| \\ &= \|DAD^{-1}Dz\| \\ &\leq \|DAD^{-1}\|\|Dz\| \end{aligned}$$

Therefore, $V(\mathbf{sat}(Az)) - V(z) \leq -(1 - \|DAD^{-1}\|)V(z)$, which satisfies the stability conditions when $\|DAD^{-1}\| < 1$.

5. *Boundaries of sublevel sets and LaSalle's theorem.* The set $\{z \in \mathbf{R}^n \mid \dot{V}(z) = 0\}$ arising in LaSalle's theorem is usually a 'thin' hypersurface, but it need not be. Carefully prove global asymptotic stability of

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 - \max\{0, x_1\} \max\{0, x_2\}.$$

(You might want to sketch out the vector field to understand the dynamics.)

Solution: Define our Lyapunov function as $V(x) = x^T x$, which is positive definite. Then

$$\begin{aligned}
\dot{V}(x) &= 2x_1\dot{x}_1 + 2x_2\dot{x}_2 \\
&= 2x_1x_2 + 2x_2\left(-x_1 - \max\{0, x_1\}\max\{0, x_2\}\right) \\
&= -2x_2\max\{0, x_1\}\max\{0, x_2\} \le 0.
\end{aligned}$$

Now we see what happens when $\dot{V}(x) = 0$:

$$\begin{aligned}
\dot{V}(x) = 0 &\Rightarrow -2x_2\max\{0, x_1\}\max\{0, x_2\} = 0 \\
&\Rightarrow x_1 \le 0 \text{ or } x_2 \le 0 \\
&\Rightarrow \left\{\begin{array}{rcr} \dot{x}_1 &=& x_2 \\ \dot{x}_2 &=& -x_1 \end{array}\right. \\
&\Rightarrow \left\{\begin{array}{rcr} x_1(t) &=& A\cos(t) + B\sin(t) \\ x_2(t) &=& B\cos(t) - A\sin(t). \end{array}\right.
\end{aligned}$$

The trace of x_1, x_2 is a circle, so if A and B are both nonzero, $x_1(t) > 0$ and $x_2(t) > 0$ for some t. This implies that the only solution of $\dot{V}(x) = 0$ is $x(t) = 0$ for all t. By LaSalle's theorem, this system is globally asymptotically stable.

6. *LQR control with quantized gain matrix.* Consider the ODE

$$\dddot{y}(t) + 0.95\ddot{y}(t) + 0.75\dot{y}(t) + 0.85y(t) = u(t),$$

with linear state feedback

$$u(t) = k_1 y(t) + k_2\dot{y}(t) + k_3\ddot{y}(t).$$

(a) Find k_1^{opt}, k_2^{opt}, and k_3^{opt} that minimize

$$J = \int_0^\infty \left(y(t)^2 + u(t)^2\right)\, dt.$$

(Of course, you must explain your method.)
What is J^{opt}, the minimum value of J, for initial condition $y(0) = \dot{y}(0) = \ddot{y}(0) = 1$?

(b) Let $k_i^{\text{quant}} = \textbf{round}(k_i^{\text{opt}})$ (where $\textbf{round}(a)$ denotes the nearest integer to a), and let J^{quant} be the value of J when $k = k^{\text{quant}}$. What is the value of J^{quant} for initial condition $y(0) = \dot{y}(0) = \ddot{y}(0) = 1$?
For what (nonzero) initial condition is the ratio $J^{\text{quant}}/J^{\text{opt}}$ maximum? What is this maximum value?

Solution:

(a) If we define $x(t) = (y(t), \dot{y}(t), \ddot{y}(t))$, we have

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t)$$

with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.85 & -0.75 & -0.95 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}.$$

This is an LQR problem with $Q = C^TC$ and $R = 1$. Defining

$$K = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix},$$

we have that $K = -B^TP$, where P is the solution of the ARE

$$C^TC + A^TP + PA - PBB^TP = 0.$$

The minimum value of J for $y(0) = \dot{y}(0) = \ddot{y}(0) = 1$ is 11.35.

(b) Let $\tilde{K}$ be the gain matrix with quantized coefficients. To find the value of J in this case, we need to solve (for P_{quant}) the Lyapunov equation

$$(A + B\tilde{K})^T P_{\text{quant}} + P_{\text{quant}}(A + B\tilde{K}) + (C^TC + \tilde{K}^T\tilde{K}) = 0.$$

We then have that $J^{\text{quant}} = x(0)^T P_{\text{quant}} x(0)$. Doing this we find that the value of J^{quant} in this case is 14.1, which is about 25% more than the minimum value.

To find the maximum value of $J^{\text{quant}}/J^{\text{opt}}$ we note that

$$\frac{J^{\text{quant}}}{J^{\text{opt}}} = \frac{x^T P_{\text{quant}} x}{x^T P x} = \frac{x^T P^{1/2} P^{-1/2} P_{\text{quant}} P^{-1/2} P^{1/2} x}{\|P^{1/2}x\|^2} = \frac{z^T T z}{\|z\|^2},$$

where $z = P^{1/2}x$ and $T = P^{-1/2} P_{\text{quant}} P^{-1/2}$. Therefore

$$\frac{J^{\text{quant}}}{J^{\text{opt}}} \leq \lambda_{\max} T$$

The initial condition which achieves this maximum can be found by finding the eigenvector corresponding to the maximum eigenvalue of T by $x(0) = P^{-1/2} v_{\max}$. For this system it turns out that

$$\frac{J^{\text{quant}}}{J^{\text{opt}}} \leq 1.37,$$

and the maximizing initial condition is

$$x(0) = (0.53, -1.74, 1.60).$$

7. *Schur complements and matrix inequalities.* Consider a matrix $X = X^T \in \mathbf{R}^{n\times n}$ partitioned as

$$X = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix},$$

where $A \in \mathbf{R}^{k\times k}$. If $\det A \neq 0$, the matrix $S = C - B^T A^{-1} B$ is called the *Schur complement* of A in X. Schur complements arise in many situations and appear in many important formulas and theorems. For example, we have $\det X = \det A \det S$. (You don't have to prove this.)

The Schur complement arises in several characterizations of positive definiteness or semidefiniteness of a block matrix. As examples we have the following three theorems:

- $X \succ 0$ if and only if $A \succ 0$ and $S \succ 0$.
- If $A \succ 0$, then $X \succeq 0$ if and only if $S \succeq 0$.
- $X \succeq 0$ if and only if $A \succeq 0$, $B^T(I - AA^\dagger) = 0$ and $C - B^T A^\dagger B \succeq 0$, where $A^\dagger$ is the pseudo-inverse of A. ($C - B^T A^\dagger B$ serves as a generalization of the Schur complement in the case where A is positive semidefinite but singular.)

Prove *one* of these theorems. (You can choose which one.)

Solution

- *If $A \succ 0$, then $X \succeq 0$ if and only if $S \succeq 0$.*
  From homework 1, we know that if $A \succ 0$, then $\inf_u f(u,v) = v^T S v$. If $S \succeq 0$, then

  $$f(u,v) \geq \inf_u f(u,v) = v^T S v \geq 0$$

  for all u, v, and hence $X \succeq 0$. This proves the 'if'-part.

  To prove the 'only if'-part we note that $f(u,v) \geq 0$ for all (u,v) implies that $\inf_u f(u,v) \geq 0$ for all v, *i.e.*, $S \succeq 0$.

- *$X \succeq 0$ if and only if $A \succeq 0$, $B^T(I - AA^\dagger) = 0$, $C - B^T A^\dagger B \succeq 0$.*
  Suppose $A \in \mathbf{R}^{k\times k}$ with $\mathbf{Rank}(A) = r$. Then there exist matrices $Q_1 \in \mathbf{R}^{k\times r}$, $Q_2 \in \mathbf{R}^{k\times(k-r)}$ and an invertible diagonal matrix $\Lambda \in \mathbf{R}^{r\times r}$ such that

  $$A = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Q_1 & Q_2 \end{bmatrix}^T,$$

  and $[Q_1\ Q_2]^T [Q_1\ Q_2] = I$. The matrix

  $$\begin{bmatrix} Q_1 & Q_2 & 0 \\ 0 & 0 & I \end{bmatrix} \in \mathbf{R}^{n\times n}$$

is nonsingular, and therefore

$$\begin{aligned}
\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \succeq 0 \quad &\Longleftrightarrow \quad \begin{bmatrix} Q_1 & Q_2 & 0 \\ 0 & 0 & I \end{bmatrix}^T \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \begin{bmatrix} Q_1 & Q_2 & 0 \\ 0 & 0 & I \end{bmatrix} \succeq 0 \\
&\Longleftrightarrow \quad \begin{bmatrix} \Lambda & 0 & Q_1^T B \\ 0 & 0 & Q_2^T B \\ B^T Q_1 & B^T Q_2 & C \end{bmatrix} \succeq 0 \\
&\Longleftrightarrow \quad Q_2^T B = 0, \begin{bmatrix} \Lambda & Q_1^T B \\ B^T Q_1 & C \end{bmatrix} \succeq 0.
\end{aligned}$$

We have $\Lambda \succ 0$ if and only if $A \succeq 0$. It can be verified that

$$A^\dagger = Q_1 \Lambda^{-1} Q_1^T, \quad I - AA^\dagger = Q_2 Q_2^T.$$

Therefore

$$Q_2^T B = 0 \Longleftrightarrow Q_2 Q_2^T B = (I - A^\dagger A) B = 0.$$

Moreover, since Λ is invertible,

$$\begin{bmatrix} \Lambda & Q_1^T B \\ B^T Q_1 & C \end{bmatrix} \succeq 0 \Longleftrightarrow \Lambda \succ 0, \quad C - B^T Q_1 \Lambda^{-1} Q_1^T B = C - B^T A^\dagger B \succeq 0.$$