EE363 Midterm Exam Solutions, May 2026

- This is a 75-minute in-class exam.
- Write your answers in the provided blue book. Make sure to clearly label each page with the problem number you are working on.
- If we ask you to "show" something is true, we are expecting a concise, intuitive derivation *not a formal proof*.
- You may use any books or paper notes.
- You may not use any electronic resources. For example, you cannot use a laptop, tablet or phone, run Julia code, search online, or consult a large language model.
- You may not discuss the exam or course material with others until everyone has taken the exam.
- If you have a question, please ask the staff. We have done our best to make the exam unambiguous. So unless there is a mistake, we are unlikely to say much.
- Please box or otherwise highlight your solution on the page.
- The problems on this exam are split into parts. Each part has a solution under 1/2 page. Thus, if you find yourself producing a complicated and time-consuming solution, stop and reconsider your assumptions.
- Good luck!

1. **Multiple choice questions (10 points).** For each of the questions below, select exactly one correct answer. All matrices have compatible dimensions.

   a) Affine systems (select one statement that is always true):

      Consider the affine system
      $$\dot{x} = Ax + b.$$

      i. An equilibrium exists if and only if $b \in \operatorname{null}(A)$.
      ii. If an equilibrium exists, it is unique if and only if A is singular.
      iii. If all eigenvalues of A have nonpositive real part, then every trajectory converges to an equilibrium.
      iv. If A is nonsingular, the unique equilibrium is $x_e = -A^{-1}b$; if, in addition, all eigenvalues of A have negative real part, then every trajectory converges to x_e.

   b) Controllability (select one condition equivalent to controllability):

      Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$.

      i. $\operatorname{rank}\begin{bmatrix} sI - A & B \end{bmatrix} = n$ for every $s \in \mathbb{C}$.
      ii. $\operatorname{rank}\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = m$.
      iii. $\operatorname{rank}\begin{bmatrix} sI - A \\ B^\mathsf{T} \end{bmatrix} = n$ for every $s \in \mathbb{C}$.
      iv. $B^\mathsf{T} v \neq 0$ for every right eigenvector v of A.

   c) Continuous-time minimum energy control (select one statement that is always true):

      Consider the continuous-time system
      $$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0,$$
      where $x_0 \neq 0$, and assume (A, B) is controllable. For $T > 0$, let $u^{(T)}$ be a minimum-energy input that drives the state to zero at time T, meaning that $x(T) = 0$ and $u^{(T)}$ minimizes
      $$E_T = \int_0^T \|u(t)\|_2^2 \, dt$$
      among all inputs that satisfy $x(T) = 0$. Which statement is always true?

      i. $E_{20} \leq E_{10}$.
      ii. $\|u^{(10)}(t)\|_2 \geq \|u^{(20)}(t)\|_2$ for every $t \in [0, 10]$.
      iii. $u^{(20)}(t) = 0$ for every $t \in [10, 20]$.
      iv. The inputs $u^{(10)}$ and $u^{(20)}$ may fail to exist even though (A, B) is controllable.

   d) Exact zero-order-hold discretization (select one statement that is always true):

      Consider the continuous-time system
      $$\dot{x}(t) = Ax(t) + Bu(t),$$
      and assume all eigenvalues of A have negative real part. Let S_h be the exact zero-order-hold discretization with sampling period $h > 0$:
      $$x_d(k+1) = A_d x_d(k) + B_d u_d(k),$$

where

$$A_d = e^{hA}, \qquad B_d = \left(\int_0^h e^{\tau A}\, d\tau\right) B.$$

Which statement is always true?

i. A_d is discrete-time stable for every $h > 0$.
ii. Every eigenvalue of A_d has negative real part.
iii. $A_d = I + hA$ for every $h > 0$.
iv. $B_d = hB$ for every $h > 0$.

e) Discretized optimal control (select one statement that is always true):

Consider the exact zero-order-hold discretization S_h from the previous part. Let N be a positive integer, let $T = Nh$, and suppose $u_d^\star(0), \ldots, u_d^\star(N-1)$ is a minimum-energy discrete-time input sequence that drives $x_d(0) = x_0$ to $x_d(N) = 0$, where discrete energy is measured as

$$h \sum_{k=0}^{N-1} \|u_d(k)\|_2^2.$$

Define the associated continuous-time zero-order-hold input by

$$u^\star(t) = u_d^\star(k), \qquad kh \le t < (k+1)h.$$

Which statement is always true?

i. $u^\star$ is necessarily a continuous-time minimum-energy input among all square-integrable inputs on $[0, T]$.
ii. The input $u^\star$ drives the original continuous-time system to $x(T) = 0$.
iii. If the sampling period is refined to $\tilde{h} = h/2$, then the $2N$-step sequence

$$\tilde{u}_d(2k) = \tilde{u}_d(2k+1) = u_d^\star(k), \qquad k = 0, \ldots, N-1,$$

must be a minimum-energy sequence for the refined discretization over the same time interval $[0, T]$.
iv. None of the above.

Solution.

a) Affine systems:

i. False. An equilibrium exists if and only if $-b \in \operatorname{range}(A)$.
ii. False. If an equilibrium exists, it is unique if and only if A is nonsingular.
iii. False. For example, purely imaginary eigenvalues can give nonconvergent trajectories.
iv. True.

b) Controllability:

i. True. This is the PBH controllability test.
ii. False. The controllability matrix must have rank n, not rank m.
iii. False. This is not the PBH controllability test in general; the vertical version would involve $sI - A^\mathsf{T}$.

iv. False. The PBH controllability test involves left eigenvectors of A, not right eigenvectors of A.

c) Continuous-time minimum energy control:

i. True. Take the 10-second optimal input $u^{(10)}$, and define a 20-second input by using $u^{(10)}$ on $[0, 10]$ and then setting the input equal to zero on $[10, 20]$. Since the state is zero at time 10 and $u = 0$ afterwards, the state remains zero until time 20. This gives a feasible 20-second input with energy E_{10}, so the minimum possible 20-second energy satisfies $E_{20} \leq E_{10}$.

ii. False. Minimum energy gives an inequality between total energies, not a pointwise inequality between the input norms.

iii. False. The 20-second optimal input need not be zero on $[10, 20]$.

iv. False. Since (A, B) is controllable, the state can be driven to zero in any positive amount of time.

d) Exact zero-order-hold discretization:

i. True. If λ_i is an eigenvalue of A, then $e^{h\lambda_i}$ is an eigenvalue of $A_d = e^{hA}$. Since $\Re(\lambda_i) < 0$, we have

$$|e^{h\lambda_i}| = e^{h\Re(\lambda_i)} < 1.$$

Therefore A_d is stable for every $h > 0$.

ii. False. Discrete-time stability requires eigenvalues to have magnitude less than one, not negative real part.

iii. False. $I + hA$ is the forward-Euler approximation to e^{hA}, not the exact zero-order-hold matrix in general.

iv. False. In general,

$$B_d = \left(\int_0^h e^{\tau A} \, d\tau \right) B,$$

which is not equal to hB except in special cases.

e) Discretized optimal control:

i. False. The discrete-time problem optimizes over inputs that are constant on each sampling interval. The continuous-time minimum-energy input is optimized over a larger class of inputs and need not be piecewise constant.

ii. True. This is exactly the defining property of the exact zero-order-hold discretization: under the zero-order-hold input $u^\star$, the sampled continuous-time state satisfies $x(kh) = x_d(k)$. Since $x_d(N) = 0$, we have $x(T) = x(Nh) = 0$.

iii. False. The repeated sequence is feasible for the refined discretization, because it represents the same continuous-time zero-order-hold input. But it need not be optimal for the refined problem, since refining the sampling period allows a larger class of piecewise-constant inputs.

iv. False.

2. State and output trajectories (10 points). Consider the autonomous continuous-time system

$$\dot{x} = Ax, \qquad y = Cx,$$

where $x(t) \in \mathbb{R}^5$, $y(t) \in \mathbb{R}$, and

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

We say that a vector-valued signal z is asymptotically 2π-periodic if

$$\|z(t+2\pi) - z(t)\| \to 0 \text{ as } t \to \infty.$$

a) Write $x_1(t), \ldots, x_5(t)$ in terms of $x_1(0), \ldots, x_5(0)$. You might find the following fact useful:

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad e^{tJ} = \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}.$$

b) Write $y(t)$ in terms of $x_1(0), \ldots, x_5(0)$.

c) Is every state trajectory $x(t)$ bounded? If not, give an exact condition on $x(0)$ under which $x(t)$ is bounded.

d) Is every state trajectory $x(t)$ asymptotically 2π-periodic? If not, give an exact condition on $x(0)$ under which $x(t)$ is asymptotically 2π-periodic.

e) Is every output trajectory $y(t)$ bounded? Does every output trajectory converge? Is every output trajectory asymptotically 2π-periodic? Briefly justify your answers.

f) Find all initial conditions $x(0)$ for which $y(t) = 0$ for all $t \geq 0$. Does this set include an initial condition whose state trajectory is unbounded? If so, give one such initial condition.

Solution. The state equations are

$$\dot{x}_1 = -x_1, \qquad \dot{x}_2 = x_3, \qquad \dot{x}_3 = 0,$$

$$\dot{x}_4 = -x_5, \qquad \dot{x}_5 = x_4.$$

a) The first three components are

$$x_1(t) = e^{-t}x_1(0), \qquad x_2(t) = x_2(0) + tx_3(0), \qquad x_3(t) = x_3(0).$$

For the last two components,

$$\ddot{x}_4 = -x_4, \qquad \ddot{x}_5 = -x_5,$$

with $\dot{x}_4(0) = -x_5(0)$ and $\dot{x}_5(0) = x_4(0)$. Hence

$$x_4(t) = x_4(0)\cos t - x_5(0)\sin t,$$

$$x_5(t) = x_4(0)\sin t + x_5(0)\cos t.$$

b) Since $y = x_1 + x_4$,

$$y(t) = e^{-t}x_1(0) + x_4(0)\cos t - x_5(0)\sin t.$$

c) Not every state trajectory is bounded because

$$x_2(t) = x_2(0) + tx_3(0).$$

The state is bounded if and only if $x_3(0) = 0$.

d) Not every state trajectory is asymptotically 2π-periodic because of $x_2(t)$. The condition for state trajectory to be asymptotically 2π-periodic is $x_3(0) = 0$. In this case the only non-periodic component is $e^{-t}x_1(0)$, which decays to zero.

e) Every output trajectory is bounded because it is the sum of a decaying exponential and a sinusoid. Not every output trajectory converges. If $x_4(0) = 1$ and all other initial components zero, $y(t) = \cos t$. Every output trajectory is asymptotically 2π-periodic, since as $t \to \infty$

$$y(t + 2\pi) - y(t) = e^{-(t+2\pi)}x_1(0) - e^{-t}x_1(0) = (e^{-2\pi} - 1)e^{-t}x_1(0) \to 0.$$

f) From the expression for $y(t)$, we have $y(t) = 0$ for all $t \geq 0$ if and only if

$$x_1(0) = 0, \qquad x_4(0) = 0, \qquad x_5(0) = 0.$$

There is no restriction on $x_2(0)$ or $x_3(0)$. These conditions allow for unbounded state trajectories by setting $x_3(0) \neq 0$. For example,

$$x(0) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

gives $y(t) = 0$ for all $t \geq 0$ but $x_2(t) = t$.

3. Minimum-energy control with coasting (10 points). Consider the discrete-time linear system

$$x(t+1) = Ax(t) + Bu(t), \qquad x(0) = 0,$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$. We would like to reach $x(N) = x_{\text{des}}$. The actuator is available only during the first $N - r$ time steps, where $1 \le r < N$, and during the last r time steps the system must coast, i.e.,

$$u(N-r) = u(N-r+1) = \cdots = u(N-1) = 0.$$

Define

$$U = \begin{bmatrix} u(N-r-1) \\ u(N-r-2) \\ \vdots \\ u(0) \end{bmatrix}.$$

The energy of the input sequence is

$$E = \sum_{t=0}^{N-r-1} \|u(t)\|_2^2 = \|U\|_2^2.$$

Do not assume A is invertible unless explicitly stated.

a) Find the matrix M_{c} such that $x(N) = M_{\text{c}}U$.

b) Give a necessary and sufficient condition on x_{des} for it to be reachable at time N under the coasting constraint. When is every $x_{\text{des}} \in \mathbb{R}^n$ reachable under this constraint?

c) Assume in this part that M_{c} has full row rank. Give an expression for the input sequence $U^\star$ that reaches x_{des} and minimizes E.

d) Under the same full-row-rank assumption, give the minimum energy $E_{\min}(x_{\text{des}})$ as a quadratic form in x_{des}.

e) Suppose $N - r \ge n$, and suppose the usual pair (A, B) is controllable. A student claims that the coasting constraint cannot hurt reachability, so every x_{des} is reachable at time N. Is the student's claim true if A is invertible? Is it true without assuming A is invertible? Give a brief justification in each case.

Solution. For the unconstrained part of the input sequence, the state at time N is

$$x(N) = \sum_{t=0}^{N-r-1} A^{N-1-t}Bu(t),$$

since the inputs from $N - r$ through $N - 1$ are zero.

a) Therefore

$$x(N) = M_{\text{c}}U,$$

where

$$M_{\text{c}} = \begin{bmatrix} A^rB & A^{r+1}B & \cdots & A^{N-1}B \end{bmatrix}.$$

b) The target x_{des} is reachable under the coasting constraint if and only if

$$x_{\text{des}} \in \text{range}(M_{\text{c}}).$$

Every target $x_{\text{des}} \in \mathbb{R}^n$ is reachable if and only if

$$\text{rank}(M_{\text{c}}) = n.$$

c) Since M_{c} has full row rank, the minimum-norm solution of $M_{\mathrm{c}}U = x_{\mathrm{des}}$ is

$$U^\star = M_{\mathrm{c}}^\top (M_{\mathrm{c}} M_{\mathrm{c}}^\top)^{-1} x_{\mathrm{des}}.$$

d) Let

$$W_{\mathrm{c}} = M_{\mathrm{c}} M_{\mathrm{c}}^\top.$$

Then

$$W_{\mathrm{c}} = \sum_{k=r}^{N-1} A^k B B^\top (A^\top)^k.$$

The minimum energy is

$$E_{\min}(x_{\mathrm{des}}) = \|U^\star\|_2^2 = x_{\mathrm{des}}^\top W_{\mathrm{c}}^{-1} x_{\mathrm{des}}.$$

e) If A is invertible, the student's claim is true. We can write

$$M_{\mathrm{c}} = A^r \begin{bmatrix} B & AB & \cdots & A^{N-r-1}B \end{bmatrix}.$$

Since A^r is invertible, multiplying by A^r does not change the rank. Because $N - r \geq n$ and (A, B) is controllable, the second matrix has rank n. Hence M_{c} has rank n, and every target is reachable.

Without assuming A is invertible, the claim is false. A scalar counterexample is

$$n = m = 1, \qquad A = 0, \qquad B = 1, \qquad N = 2, \qquad r = 1.$$

The pair (A, B) is controllable because the controllability matrix is just $[B] = [1]$. Also $N - r = 1 \geq n$. But with coasting,

$$M_{\mathrm{c}} = A^r B = AB = 0,$$

so $x(2) = 0$ for every allowable input $u(0)$. Thus any nonzero x_{des} is not reachable.

4. A realization and its transfer function (10 points). Consider the continuous-time system

$$\dot{x} = Ax + Bu, \qquad y = Cx,$$

where

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}.$$

For this problem, call a realization internally stable if all eigenvalues of its A matrix have negative real part. Call a transfer function stable if all of its poles have negative real part.

a) Compute the transfer function $G(s) = C(sI - A)^{-1}B$.

b) For each eigenvalue of A, say whether the associated mode is controllable and whether the associated mode is observable. Briefly justify your answer.

c) Which eigenvalues of A appear as poles of $G(s)$? Briefly explain any discrepancy between the eigenvalues of A and the poles of $G(s)$.

d) Is the realization given in the problem internally stable? Is $G(s)$ stable? Briefly explain.

e) Assume $u(t) = 0$. Find all initial conditions $x(0)$ for which $y(t) = 0$ for all $t \geq 0$. Does this set contain a nonzero initial condition for which $\|x(t)\| \to \infty$? If so, give one.

f) Give a realization of the same transfer function with the smallest possible state dimension. Briefly justify why the dimension is smallest possible.

Solution.

a) The transfer function is given by

$$G(s) = C(sI - A)^{-1}B.$$

Since A is diagonal,

$$(sI - A)^{-1} = \begin{bmatrix} \frac{1}{s+1} & 0 & 0 \\ 0 & \frac{1}{s-2} & 0 \\ 0 & 0 & \frac{1}{s+3} \end{bmatrix}.$$

Therefore

$$G(s) = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{s+1} & 0 & 0 \\ 0 & \frac{1}{s-2} & 0 \\ 0 & 0 & \frac{1}{s+3} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{s+1}.$$

b) The eigenvalues of A are -1, 2, and -3, with corresponding eigenvectors e_1, e_2, and e_3. Controllability and observability of e_i can be deduced by simply looking at the corresponding entries of B and C. Thus:

eigenvalue	controllable?	observable?
-1	yes	yes
2	yes	no
-3	no	yes

c) The only pole of $G(s)$ is $s = -1$. The eigenvalue 2 does not appear as a pole because its mode is unobservable, and the eigenvalue -3 does not appear as a pole because its mode is uncontrollable.

d) The given realization is not internally stable, since A has eigenvalue 2. The transfer function is stable, since its only pole is -1, which has negative real part. Thus the input-output behavior from zero initial condition is stable, even though the realization has an unstable internal mode.

e) With $u(t) = 0$,

$$x(t) = \begin{bmatrix} e^{-t}x_1(0) \\ e^{2t}x_2(0) \\ e^{-3t}x_3(0) \end{bmatrix}, \qquad y(t) = e^{-t}x_1(0) + e^{-3t}x_3(0).$$

Since e^{-t} and e^{-3t} are linearly independent functions, $y(t) = 0$ for all $t \geq 0$ if and only if

$$x_1(0) = 0, \qquad x_3(0) = 0.$$

Thus the set of such initial conditions is

$$\left\{ \begin{bmatrix} 0 \\ \alpha \\ 0 \end{bmatrix} : \alpha \in \mathbb{R} \right\}.$$

This set contains nonzero initial conditions with unbounded state trajectories. For example, with $x(0) = e_2$ and $u(t) = 0$, we have $x(t) = e^{2t}e_2$, so $\|x(t)\| \to \infty$, while $y(t) = 0$ for all $t \geq 0$.

f) One smallest-dimension realization is

$$A_m = [-1], \qquad B_m = [1], \qquad C_m = [1].$$

It realizes

$$C_m(sI - A_m)^{-1}B_m = \frac{1}{s+1}.$$

The dimension cannot be zero since that would give a constant transfer function ($G(s) = D$), therefore 1 is the smallest possible dimension.