EE363 Homework 1 Solutions

Spring 2026

2.20. State equations for a linear mechanical system. The equations of motion of a lumped mechanical system undergoing small motions can be expressed as

$$M\ddot{q} + D\dot{q} + Kq = f$$

where $q(t) \in \mathbb{R}^k$ is the vector of deflections, M, D, and K are the *mass*, *damping*, and *stiffness* matrices, respectively, and $f(t) \in \mathbb{R}^k$ is the vector of externally applied forces. Assuming M is invertible, write linear system equations for the mechanical system, with state

$$x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix},$$

input $u = f$, and output $y = q$.

Solution. We need to express the output q and the state derivative, $\dot{q}$ and $\ddot{q}$, as a linear function of the state variables q, $\dot{q}$ and the input f. In other words, we should find matrices A, B, C and D such that

$$\begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix} = A \begin{bmatrix} q \\ \dot{q} \end{bmatrix} + Bf, \quad q = C \begin{bmatrix} q \\ \dot{q} \end{bmatrix} + Df.$$

Matrices C and D are easy to find: simply, for the second equation to hold we should have

$$C = \begin{bmatrix} I & 0 \end{bmatrix}, \quad D = 0.$$

A and B are a bit harder to find. We will use the differential equation to express $\ddot{q}$ in terms of q, $\dot{q}$ and f. From the given dynamics equation $M\ddot{q} + D\dot{q} + Kq = f$, and assuming M is invertible, we get

$$\ddot{q} = -M^{-1}Kq - M^{-1}D\dot{q} + M^{-1}f,$$

which expresses $\ddot{q}$ in terms of q, $\dot{q}$, and f. Now we can write the linear dynamical system equations for the system. In block matrix notation we have

$$\frac{d}{dt}\begin{bmatrix} q \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix} \begin{bmatrix} q \\ \dot{q} \end{bmatrix} + \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix} u, \quad y = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} q \\ \dot{q} \end{bmatrix},$$

so the matrices in linear dynamical system description are:

$$A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix}, \quad C = \begin{bmatrix} I & 0 \end{bmatrix}, \quad D = 0.$$

2.70. Matrix representation of linear systems. Consider the (discrete-time) linear dynamical system

$$x(t+1) = A(t)x(t) + B(t)u(t), \quad y(t) = C(t)x(t) + D(t)u(t).$$

Find a matrix G such that

$$\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(N) \end{bmatrix} = G \begin{bmatrix} x(0) \\ u(0) \\ \vdots \\ u(N) \end{bmatrix}.$$

The matrix G shows how the output at $t = 0, \ldots, N$ depends on the initial state $x(0)$ and the sequence of inputs $u(0), \ldots, u(N)$.

Solution. For $t = 0$, $x(t+1) = A(t)x(t) + B(t)u(t)$ becomes

$$x(1) = A(0)x(0) + B(0)u(0).$$

For $t = 1$

$$\begin{aligned} x(2) &= A(1)x(1) + B(1)u(1) \\ &= A(1)\left[A(0)x(0) + B(0)u(0)\right] + B(1)u(1) \\ &= A(1)A(0)x(0) + A(1)B(0)u(0) + B(1)u(1). \end{aligned}$$

For $t = 2$

$$\begin{aligned} x(3) &= A(2)x(2) + B(2)u(2) \\ &= A(2)\left[A(1)A(0)x(0) + A(1)B(0)u(0) + B(1)u(1)\right] + B(2)u(2) \\ &= A(2)A(1)A(0)x(0) + A(2)A(1)B(0)u(0) + A(2)B(1)u(1) + B(2)u(2). \end{aligned}$$

Now it is easy to guess the general expression for $x(i)$ in terms of $x(0), u(0), \ldots, u(i-1)$ (which can be later proved inductively) as follows

$$\begin{aligned} x(i) = {} & A(i-1)A(i-2)\cdots A(0)x(0) \\ & + A(i-1)A(i-2)\cdots A(1)B(0)u(0) \\ & + A(i-1)A(i-2)\cdots A(2)B(1)u(1) \\ & + A(i-1)A(i-2)\cdots A(3)B(2)u(2) \\ & \vdots \\ & + A(i-1)B(i-2)u(i-2) \\ & + B(i-1)u(i-1). \end{aligned}$$

Given $x(i)$ as above, $y(i)$ is simply found from $y(i) = C(i)x(i) + D(i)u(i)$ in terms of $x(0), u(0), \ldots, u(i)$ as

$$\begin{aligned} y(i) = {} & C(i)A(i-1)A(i-2)\cdots A(0)x(0) \\ & + C(i)A(i-1)A(i-2)\cdots A(1)B(0)u(0) \\ & + C(i)A(i-1)A(i-2)\cdots A(2)B(1)u(1) \\ & + C(i)A(i-1)A(i-2)\cdots A(3)B(2)u(2) \\ & \vdots \\ & + C(i)A(i-1)B(i-2)u(i-2) \\ & + C(i)B(i-1)u(i-1) \\ & + D(i)u(i). \end{aligned}$$

Therefore

$$G_{i1} = \begin{cases} C(0) & \text{if } i = 1 \\ C(i-1)A(i-2)\cdots A(0) & \text{if } 2 \le i \le N+1 \end{cases}$$

and for $i = 1, 2, \ldots, N+1$

$$G_{ij} = \begin{cases} 0 & \text{if } N+2 \ge j > i+1 \\ D(i-1) & \text{if } j = i+1 \\ C(i-1)B(i-2) & \text{if } j = i \\ C(i-1)A(i-2)\cdots A(j-1)B(j-2) & \text{if } 2 \le j < i \end{cases}$$

Note: A number of students used product notation "$\prod_{k=1}^{n} A_k$" to denote the product $A_n \cdots A_1$. It's advisable to avoid this notation with matrices, since it's not clear (unless you specify explicitly) whether this means that or $A_1 \cdots A_n$, and the two aren't equal, because matrix multiplication is not commutative.

9.1460. Linear dynamical system with constant input. We consider the system $\dot{x} = Ax + b$, with $x(t) \in \mathbb{R}^n$. A vector x_e is an equilibrium point if $0 = Ax_e + b$. (This means that the constant trajectory $x(t) = x_e$ is a solution of $\dot{x} = Ax + b$.)

a) When is there an equilibrium point?

b) When are there multiple equilibrium points?

c) When is there a unique equilibrium point?

d) Now suppose that x_e is an equilibrium point. Define $z(t) = x(t) - x_e$. Show that $\dot{z} = Az$. From this, give a general formula for $x(t)$ (involving x_e, $\exp(tA)$, $x(0)$).

e) Show that if all eigenvalues of A have negative real part, then there is exactly one equilibrium point x_e, and for any trajectory $x(t)$, we have $x(t) \to x_e$ as $t \to \infty$.

Solution.

a) An equilibrium point x_e exists if and only if $0 = Ax_e + b$, *i.e.*, $-b = Ax_e$. This happens exactly when $-b \in \text{range}(A)$.

b) If x_e is any equilibrium point, and $z \in \text{null}(A)$, then $x_e + z$ is also an equilibrium point. It follows that in order to have multiple equilibrium points, we need $\text{null}(A) \ne \{0\}$, as well as $b \in \text{range}(A)$.

c) For uniqueness, we need that $\text{null}(A) = \{0\}$, in addition to $b \in \text{range}(A)$. The nullspace condition implies that A is nonsingular. But this means that $\text{range}(A) = \mathbb{R}^n$, so the condition $b \in \text{range}(A)$ holds automatically. In this case, the unique equilibrium point is $x_e = -A^{-1}b$. In summary: there is a unique equilibrium point if and only if A is nonsingular; in this case, we have $x_e = -A^{-1}b$.

d) $z(t) = \exp(tA)z(0)$, so

$$x(t) = x_e + \exp(tA)(x(0) - x_e).$$

e) Assume that all eigenvalues of A have negative real part. In particular, no eigenvalue can be zero, which means A is nonsingular. Therefore the unique equilibrium point is $x_e = -A^{-1}b$. Since all eigenvalues of A have negative real part, the matrix $\exp(tA)$ goes to zero as $t \to \infty$. From the formula for $x(t)$ above, we see that $x(t)$ converges to x_e.

11.1740. Another formula for the matrix exponential. You might remember that for any complex number $a \in \mathbb{C}$, $e^a = \lim_{k\to\infty}(1 + a/k)^k$. You will establish the matrix analog: for any $A \in \mathbb{R}^{n\times n}$,

$$e^A = \lim_{k\to\infty}(I + A/k)^k.$$

To simplify things, you can assume A is diagonalizable. *Hint:* diagonalize.

Solution. Assuming $A \in \mathbb{R}^{k\times k}$ is diagonalizable, there exists an invertible matrix $T \in \mathbb{R}^{n\times n}$ such that $A = T\,\mathrm{diag}(\lambda_1, \dots, \lambda_n)T^{-1}$ where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A. Therefore

$$\begin{aligned}(I + A/k)^k &= (TT^{-1} + T\,\mathrm{diag}(\lambda_1/k, \dots, \lambda_n/k)T^{-1})^k \\ &= \big(T(I + \mathrm{diag}(\lambda_1/k, \dots, \lambda_n/k))T^{-1}\big)^k \\ &= T(I + \mathrm{diag}(\lambda_1/k, \dots, \lambda_n/k))^k T^{-1}.\end{aligned}$$

But $(I + \mathrm{diag}(\lambda_1/k, \dots, \lambda_n/k))$ is diagonal and therefore its kth power is simply a diagonal matrix with diagonal entries equal to the kth power of the diagonal entries of $(I + \mathrm{diag}(\lambda_1/k, \dots, \lambda_n/k))$. Thus

$$(I + A/k)^k = T\,\mathrm{diag}((1 + \lambda_1/k)^k, \dots, (1 + \lambda_n/k)^k)T^{-1}$$

and taking the limit as $k \to \infty$ gives

$$\begin{aligned}\lim_{k\to\infty}(I + A/k)^k &= \lim_{k\to\infty} T\,\mathrm{diag}((1 + \lambda_1/k)^k, \dots, (1 + \lambda_n/k)^k)T^{-1} \\ &= T\,\mathrm{diag}(\lim_{k\to\infty}(1 + \lambda_1/k)^k, \dots, \lim_{k\to\infty}(1 + \lambda_n/k)^k)T^{-1} \\ &= T\,\mathrm{diag}(e^{\lambda_1}, \dots, e^{\lambda_n})T^{-1} \\ &= e^A,\end{aligned}$$

and we are done.

12.1920. Jordan form of a block matrix. We consider the block 2×2 matrix

$$C = \begin{bmatrix} A & I \\ 0 & A \end{bmatrix}.$$

Here $A \in \mathbb{R}^{n\times n}$, and is diagonalizable, with real, distinct eigenvalues $\lambda_1, \dots, \lambda_n$. We'll let $v_1, \dots, v_n$ denote (independent) eigenvectors of A associated with $\lambda_1, \dots, \lambda_n$.

a) Find the Jordan form J of C. Be sure to explicitly describe its block sizes.

b) Find a matrix T such that $J = T^{-1}CT$.

Solution. The eigenvalues of C are

$$\lambda_1, \lambda_1, \lambda_2, \lambda_2, \ldots, \lambda_n, \lambda_n,$$

i.e., each of the eigenvalues of A, given twice. Each λ_i, then, is either associated with two 1×1 Jordan blocks in C, or with one 2×2 Jordan block in C. It turns out that only the second possibility can occur. In other words, the Jordan form of C is block diagonal, with n 2×2 Jordan blocks, each associated with one λ_i:

$$J = \begin{bmatrix} \lambda_1 & 1 & & & & & \\ & \lambda_1 & & & & & \\ & & \lambda_2 & 1 & & & \\ & & & \lambda_2 & & & \\ & & & & \ddots & & \\ & & & & & \lambda_n & 1 \\ & & & & & & \lambda_n \end{bmatrix}.$$

This can be discovered several ways, but in any case below we'll give a matrix T such $T^{-1}AT$ has this form, which ends the discussion. This Jordan form isn't unexpected. After all, C looks like a sort of 2×2 matrix Jordan block. Let's move on to part (b). We need to find a set of independent generalized eigenvectors for C, *i.e.*, the columns of T, which we'll call $t_1, \ldots, t_{2n}$. They satisfy the equations

$$Ct_{2i-1} = \lambda_i t_{2i-1}, \quad i = 1, \ldots, n,$$

which states that the odd index t_i's are true eigenvectors of C, and the generalized eigenvector equations,

$$Ct_{2i} = t_{2i-1} + \lambda_i t_{2i}, \quad i = 1, \ldots, n.$$

Let's first start with the odd indices, which correspond to true eigenvectors of C. We first note that $Av_i = \lambda_i v_i$, so we have

$$\begin{bmatrix} A & B \\ 0 & A \end{bmatrix} \begin{bmatrix} v_i \\ 0 \end{bmatrix} = \lambda_i \begin{bmatrix} v_i \\ 0 \end{bmatrix}.$$

Thus, we can take

$$t_{2i-1} = \begin{bmatrix} v_i \\ 0 \end{bmatrix}, \quad i = 1, \ldots, n.$$

Let's find the even index generalized eigenvectors, $t_2, t_4, \ldots, t_{2n}$. They satisfy the equations

$$Ct_{2i} = t_{2i-1} + \lambda_i t_{2i}, \quad i = 1, \ldots, n.$$

To find what t_{2i} must be, we express it as $t_{2i} = [u^\mathsf{T} \; w^\mathsf{T}]^\mathsf{T}$:

$$Ct_{2i} = \begin{bmatrix} A & I \\ 0 & A \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix} = \begin{bmatrix} Au + w \\ Aw \end{bmatrix} = t_{2i-1} + \lambda_i t_{2i} = \begin{bmatrix} v_i + \lambda_i u \\ \lambda_i w \end{bmatrix}.$$

From the bottom entries, we see that $Aw = \lambda_i w$, so must take $w = v_i$ (or some multiple of v_i). Plugging this in to the top equations yields $Au + v_i = v_i + \lambda_i u$, from which we conclude $u = v_i$ as well. Thus, we can take as even generalized eigenvectors

$$t_{2i} = \begin{bmatrix} v_i \\ 0 \end{bmatrix}, \quad i = 1, \ldots, n.$$

Let's find the even index generalized eigenvectors, $t_2, t_4, \ldots, t_{2n}$. They satisfy the equations

$$Ct_{2i} = t_{2i-1} + \lambda_i t_{2i}, \quad i = 1, \ldots, n.$$

To find what t_{2i} must be, we express it as $t_{2i} = [u^\mathsf{T} \; w^\mathsf{T}]^\mathsf{T}$:

$$Ct_{2i} = \begin{bmatrix} A & I \\ 0 & A \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix} = \begin{bmatrix} Au + w \\ Aw \end{bmatrix} = t_{2i-1} + \lambda_i t_{2i} = \begin{bmatrix} v_i + \lambda_i u \\ \lambda_i w \end{bmatrix}.$$

From the bottom entry, we see that $Aw = \lambda_i w$, so must take $w = v_i$ (or some multiple of v_i). Plugging this in to the top entry yields $Au + v_i = v_i + \lambda_i u$, from which we conclude $u = v_i$ as well. Thus, we can take as even generalized eigenvectors

$$t_{2i} = \begin{bmatrix} v_i \\ v_i \end{bmatrix}, \quad i = 1, \ldots, n.$$

We're done! By our construction we have $T^{-1}CT = J$, where the columns of T are given above. In particular, we have

$$T = \begin{bmatrix} v_1 & v_1 & v_2 & v_2 & \cdots & v_n & v_n \\ 0 & v_1 & 0 & v_2 & \cdots & 0 & v_n \end{bmatrix}.$$