EE363 Homework 1 Solutions

Spring 2026

11.1780. Companion matrices. A matrix A of the form

$$A = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

is said to be a (top) *companion matrix*. There can be four forms of companion matrices depending on whether the a_i's occur in the first or last row, or first or last column. These are referred to as top-, bottom-, left-, or right-companion matrices. Let $\dot{x} = Ax$ where A is top-campanion.

a) Draw a block diagram for the system $\dot{x} = Ax$.

b) Find the characteristic polynomial of the system using the block diagram and show that A is nonsingular if and only if $a_n \neq 0$.

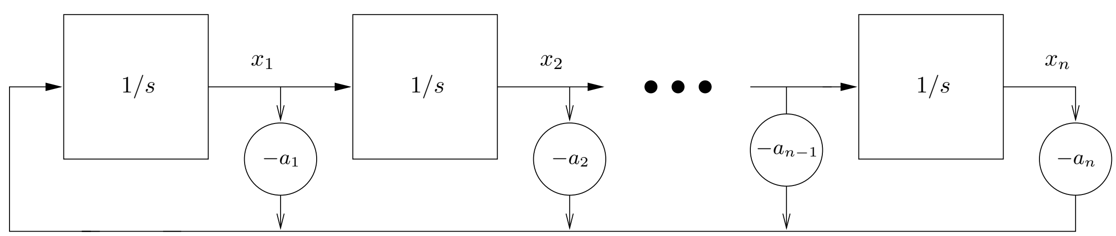
c) Show that if A is nonsingular, then A^{-1} is a bottom-companion matrix with last row $-[1\ a_1\ \cdots\ a_{n-1}]/a_n$.

d) Find the eigenvector of A associated with the eigenvalue λ.

e) Suppose that A has distinct eigenvalues $\lambda_1, \ldots, \lambda_n$. Find T such that $T^{-1}AT$ is diagonal.

Solution.

a) The block diagram of the system is:



b) Using the block diagram we can simply write a differential equation for x_n. Note that $x_{n-1} = \dot{x}_n$, $x_{n-2} = \ddot{x}_n$, $\ldots$, $x_1 = x_n^{(n-1)}$ and therefore

$$x_n^{(n)} = -a_1 x_n^{(n-1)} - a_2 x_n^{(n-2)} - \cdots - a_n x_n$$

or

$$x_n^{(n)} + a_1 x_n^{(n-1)} + a_2 x_n^{(n-2)} + \cdots + a_n x_n = 0.$$

This is an nth degree differential equation in x_n. Since all n modes of this differential equation are modes of $\dot{x} = Ax$ (which has n modes), then the modes of this differential equation are exacly the same as the modes of the original system $\dot{x} = Ax$. Therefore the characteristic polynomial is

$$s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n.$$

A is nonsingular if and only if $\det A \neq 0$ or $\det(sI - A)$ is nonzero for $s = 0$. But $\det(sI - A)|_{s=0} = a_n$ and therefore A is nonsingular if and only if $a_n \neq 0$.

c) We simply verify this:

$$\begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -1/a_n & -a_1/a_n & \cdots & -a_{n-2}/a_n & -a_{n-1}/a_n \end{bmatrix} =$$

$$\begin{bmatrix} -a_n(-1/a_n) & -a_1 + a_1 & \cdots & -a_{n-2} + a_{n-2} & -a_{n-1} + a_{n-1} \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} = I.$$

Note that since A^{-1} exists we have $a_n \neq 0$.

d) Suppose $x = [x_1 \; x_2 \cdots \; x_n]^\mathsf{T}$ is the eigenvector associated with λ so that $Ax = \lambda x$ or

$$\begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

Therefore

$$\begin{aligned} -a_1x_1 - a_2x_2 - \cdots - a_{n-1}x_{n-1} - a_nx_n &= \lambda x_1 \\ x_1 &= \lambda x_2 \\ x_2 &= \lambda x_3 \\ &\vdots \\ x_{n-1} &= \lambda x_n \end{aligned}$$

The last $n - 1$ equations yield

$$x_{n-1} = \lambda x_n, \; x_{n-2} = \lambda^2 x_n, \ldots, \; x_1 = \lambda^{n-1} x_n.$$

We have one degree of freedom in choosing the eigenvector x. Let's pick $x_n = 1$. Therefore

$$x_{n-1} = \lambda, \; x_{n-2} = \lambda^2, \ldots, x_1 = \lambda^{n-1}.$$

With these values the first equation becomes

$$-a_1\lambda^{n-1} - a_2\lambda^{n-2} - \cdots - a_{n-1}\lambda - a_n = \lambda^n$$

which is nothing but $\det(\lambda I - A) = 0$ or that λ is an eigenvalue of A. Therefore the eigenvector corresponding to the eigenvalue λ is

$$x = \begin{bmatrix} \lambda^{n-1} \\ \lambda^{n-2} \\ \vdots \\ \lambda \\ 1 \end{bmatrix}.$$

e) From part (d) we know that the eigenvector corresponding to λ_i is $[\lambda_i^{n-1}\ \lambda_i^{n-2}\ \cdots\ \lambda_i\ 1]^\mathsf{T}$. Thus,

$$T = \begin{bmatrix} \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \\ \lambda_1^{n-2} & \lambda_2^{n-2} & \cdots & \lambda_n^{n-2} \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}$$

We know that the eigenvalues are distinct and therefore the eigenvectors are independent and T is invertible. It can also be shown that

$$\det T = (-1)^{\frac{n(n-1)}{2}} \prod_{i<j} (\lambda_j - \lambda_i).$$

Since $\lambda_i \neq \lambda_j$ for $i \neq j$ we have $\det T \neq 0$. (Note: any matrix of the form

$$T = \begin{bmatrix} \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \\ \lambda_1^{n-2} & \lambda_2^{n-2} & \cdots & \lambda_n^{n-2} \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}$$

is called a *Vandermonde matrix*. A Vandermonde matrix is nonsingular if and only if $\lambda_i \neq \lambda_j$ for $i \neq j$.)

12.1910. Asymptotically periodic trajectories. We say that $x : \mathbb{R}_+ \to \mathbb{R}^n$ is *asymptotically T-periodic* if $\|x(t+T) - x(t)\|$ converges to 0 as $t \to \infty$. (We assume $T > 0$ is fixed.) Now consider the (time-invariant) linear dynamical system $\dot{x} = Ax$, where $A \in \mathbb{R}^{n \times n}$. Describe the precise conditions on A under which *all* trajectories of $\dot{x} = Ax$ are asymptotically T-periodic. Give your answer in terms of the Jordan form of A. (The period T can appear in your answer.) Make sure your answer works for 'silly' cases like $A = 0$ (for which all trajectories are constant, hence asymptotically T-periodic), or stable systems (for which all trajectories converge to 0, hence are asymptotically T-periodic). Mark your answer clearly, to isolate it from any (brief) discussion or explanation. You do not need to formally prove your answer; a brief explanation will suffice.

Solution. First note that

$$x(t+T) - x(t) = e^{A(t+T)}x(0) - e^{At}x(0) = e^{At}(e^{AT} - I)x(0)$$

so in order to have $\|x(t+T) - x(t)\| \to 0$ as $t \to 0$, for all possible $x(0)$, all elements of the matrix $e^{At}(e^{AT} - I)$ must converge to zero. Since these matrices commute we could also express this as $(e^{AT} - I)e^{At}$. Writing A in the Jordan form as $A = UJU^{-1}$, we get

$$\begin{aligned} e^{UJU^{-1}t}(e^{UJU^{-1}T} - I) &= Ue^{Jt}U^{-1}(Ue^{JT}U^{-1} - I) \\ &= Ue^{J}tU^{-1}U(e^{JT} - I)U^{-1} \\ &= Ue^{Jt}(e^{JT} - I)U^{-1} \end{aligned}$$

The matrix $e^{Jt}(e^{JT} - I)$ is block diagonal, where the ith block is $(e^{J_iT} - I)e^{J_it}$, and J_i is the ith Jordan block, with eigenvalue λ_i. Note that we do not assume the eigenvalues are distinct: we can have several Jordan blocks with the same eigenvalue. The condition is that $e^{J_it}(e^{J_iT} - I) \to 0$, as $t \to 0$, for $i = 1, \ldots, k$, where k is the number of Jordan blocks of A. If an eigenvalue has a negative real part (*e.g.*, $\Re(\lambda_i) \leq 0$), then the term e^{J_it} will converge to zero (this is regardless of the size of the block, so we can have Jordan blocks of any size for these eigenvalues). These are the stable modes of the system, and will die out as $t \to \infty$. But if $\Re(\lambda_i) > 0$, the term e^{J_it} will grow, so we can't have the condition $(e^{J_iT} - I)e^{J_it} \to 0$. Hence we can't have any eigenvalues with positive real part. A similar argument shows that for pure imaginary eigenvalues, we cannot have Jordan blocks of size greater than one; if we do, we end up with a term such as $t^pe^{\lambda t}$ where λ is pure imaginary and $p \geq 1$. Finally we consider pure imaginary eigenvalues that correspond to 1×1 Jordan blocks. These must satisfy $\lim_{t\to\infty}(e^{\lambda_iT} - 1)e^{\lambda_it} = 0$, which can only happen if $e^{\lambda_iT} = 1$, or $\lambda_i = \pm j2k\pi/T$. This makes sense; oscillatory modes with period T are allowed. The integer k takes care of oscillatory modes with a period that is a submultiple of T. In summary, the precise conditions on A are:

- all eigenvalues must satisfy $\Re(\lambda_i) \leq 0$.
- and if $\Re\lambda_i = 0$, then the associated Jordan block must have size 1, and λ must have the form $\lambda = j2m\pi/T$ for some $m \in \mathbb{Z}$.

Note that eigenvalues can be repeated, even eigenvalues with zero real part. But eigenvalues with zero real part must be associated with Jordan blocks of size 1×1. Also note that when an eigenvalue has negative real part, its Jordan block size and the imaginary part are irrelevant. We saw lots of variations that were close, but wrong, or at the least, confused. One common (and wrong) answer was that either all eigenvalues of A have negative real part (*i.e.*, $\dot{x} = Ax$ is stable) or, all the eigenvalues are pure imaginary with the form given above. This isn't correct: you can 'mix' the two conditions, block by block, in the Jordan form.

12.1930. Properties of trajectories. For each of the following statements, give the exact (necessary and sufficient) conditions on $A \in \mathbb{R}^{n\times n}$ under which the statement holds.

a) Every trajectory of $\dot{x} = Ax$ converges as $t \to \infty$. This means that, for any $x(0)$, $x(t)$ converges to some value, which need not be zero (and can depend on $x(0)$ and A).

b) Every trajectory of $\dot{x} = Ax$ is bounded. This means that, for any $x(0)$, there is an M (that can depend on $x(0)$ and A) for which $\|x(t)\| \leq M$ for all $t \geq 0$.

Your answers can refer to any concepts used in the course (eigenvalues, singular values, Jordan form, least-squares, range, nullspace, ...). We will deduct points from answers that are technically correct, but more complicated than they need to be. *You may not make any assumptions about* A (*e.g.*, that it is nonsingular, diagonalizable, etc.).

Please give only your final answer; we do not want any justification or discussion. Your answers should have a form similar to "The property in part (a) occurs if and only if all singular values of A are less than one, and A has no real eigenvalues". (This is *not* the correct answer; it is only as an example of what your answer should look like.)

Solution.

a) *The property in part (a) occurs if and only if all eigenvalues of* A *either have negative real part or are zero, and each zero eigenvalue is associated with a Jordan block of size* 1×1. The latter condition can be stated several other ways, *e.g.*, the dimension of the nullspace of A is equal to the number of zero eigenvalues of A.

   Now let's justify the condition. (We did not ask you to give any justification.)

   First suppose our condition holds. Then every solution of $\dot{x} = Ax$ is a linear combination of terms with decaying exponentials (possibly oscillatory, and including powers of t as well), associated with the eigenvalues with negative real part, and terms which are constant, associated with the zero eigenvalues. (Note that we cannot have terms that grow like t, since these would be associated with zero eigenvalues with Jordan blocks of size 2×2 or larger.) Thus, every trajectory converges. In fact, every trajectory converges to an element of the nullspace of A. (These are the equilibrium points of $\dot{x} = Ax$.)

   Now suppose our condition does not hold. This happens if A has an eigenvalue with nonnegative real part that is not zero, or a zero eigenvalue associated with a Jordan block of size 2×2 or bigger. Consider the first case. If an eigenvalue λ (associated with eigenvector v) has a positive real part, then we can find a diverging (and possibly oscillating, if λ is complex) trajectory, which therefore doesn't converge. If the real part of λ is zero, but the eigenvalue is different from zero, then it must be pure imaginary, so we can find an oscillating trajectory, which doesn't converge. Finally, suppose that A has an eigenvalue 0 associated with a Jordan block of size 2×2 or bigger. In this case we can find a trajectory of the form $x(t) = a + bt$, with $b \neq 0$, which clearly doesn't converge.

b) *The property in part (b) occurs if and only if all eigenvalues of* A *have real part less than or equal to zero, and those with real part zero (i.e., the zero and pure imaginary ones) are associated with Jordan blocks of size* 1×1. We consider the Jordan decomposition $A = TJT^{-1}$, and the resulting change of coordinates for the dynamical system: $y = T^{-1}x$. We see that the original system, represented in the new coordinates, is

$$\dot{y} = T^{-1}ATy = Jy.$$

   Since T is fixed and invertible, every trajectory of x is bounded if and only if every trajectory of y is bounded. We have that

$$y(t) = e^{tJ}y(0),$$

and we see that the trajectory of $y(t)$ is bounded for all $y(0)$ if and only if $\|e^{tJ}\|$ is bounded as t varies.

However, J has block diagonal structure $J = \mathrm{diag}(J_1, \dots, J_q)$, where J_i is the Jordan block corresponding to eigenvalue λ_i. Since J is block diagonal, $\|e^{tJ}\|$ is bounded if and only if $\|e^{tJ_i}\|$ is bounded for each i. Recall that if J_i is a Jordan block of size $k \times k$, then its exponential is given by

$$e^{tJ_i} = e^{t\lambda_i} \begin{bmatrix} 1 & t & \cdots & t^{k-1}/(k-1)! \\ & 1 & \cdots & t^{k-2}/(k-2)! \\ & & \ddots & \vdots \\ & & & 1 \end{bmatrix}.$$

We are now in a position to confirm the condition that was proposed.

If any eigenvalue λ_i of A has positive real part, then $e^{t\lambda_i}$ grows without bound, which implies that $\|e^{tJ_i}\|$ is also unbounded, and so there is an unbounded trajectory of x.

If an eigenvalue λ_i of A has negative real part, then $e^{t\lambda_i}t^p \to 0$ as $t \to \infty$ for any positive power of p, which implies that $\|e^{tJ_i}\| \to 0$, which tells us that $\|e^{tJ_i}\|$ is bounded.

If an eigenvalue λ_i has zero real part, then

$$\|e^{tJ_i}\| = \left\| e^{t\lambda_i} \begin{bmatrix} 1 & t & \cdots & t^{k-1}/(k-1)! \\ & 1 & \cdots & t^{k-2}/(k-2)! \\ & & \ddots & \vdots \\ & & & 1 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 1 & t & \cdots & t^{k-1}/(k-1)! \\ & 1 & \cdots & t^{k-2}/(k-2)! \\ & & \ddots & \vdots \\ & & & 1 \end{bmatrix} \right\|,$$

since $|e^{t\lambda_i}| = 1$. The t^p terms grow without bound, so $\|e^{tJ_i}\|$ is unbounded if the Jordan block has size $k \times k$, for $k > 1$. On the other hand, if $k = 1$, then the matrix is constant, and thus $\|e^{tJ_i}\|$ is bounded.

Thus, we see that if our condition holds, then every trajectory of x is bounded. Conversely, if every trajectory is bounded, then the condition must hold, because if not, we would be able to find a trajectory that is unbounded, which is a contradiction.

13.2000. Inverse of a linear system. Suppose $H(s) = C(sI - A)^{-1}B + D$, where D is square and invertible. You will find a linear system with transfer function $H(s)^{-1}$.

a) Start with $\dot{x} = Ax + Bu$, $y = Cx + Du$, and solve for $\dot{x}$ and u in terms of x and y. Your answer will have the form: $\dot{x} = Ex + Fy$, $u = Gx + Hy$. Interpret the result as a linear system with state x, input y, and output u.

b) Verify that

$$(G(sI - E)^{-1}F + H)(C(sI - A)^{-1}B + D) = I.$$

Hint: use the following "resolvent identity:"

$$(sI - X)^{-1} - (sI - Y)^{-1} = (sI - X)^{-1}(X - Y)(sI - Y)^{-1}$$

which can be verified by multiplying by $sI - X$ on the left and $sI - Y$ on the right.

Solution.

a) Since D^{-1} exists,

$$y = Cx + Du \quad \Longrightarrow \quad u = -D^{-1}Cx + D^{-1}y,$$

which yields

$$\dot{x} = Ax + Bu = Ax - BD^{-1}Cx + BD^{-1}y = (A - BD^{-1}C)x + BD^{-1}y.$$

Therefore,

$$\dot{x} = \left(A - BD^{-1}C\right)x + BD^{-1}y, \quad u = -D^{-1}Cx + D^{-1}y$$

and so

$$E = A - BD^{-1}C, \quad F = BD^{-1}, \quad G = -D^{-1}C, \quad H = D^{-1}.$$

So (E, F, G, H) is a realization of the inverse transfer function with the same state, input y, and output u.

b) Verifying the inverse relationship,

$$\begin{aligned}
&\left[G(sI - E)^{-1}F + H\right]\left[C(sI - A)^{-1}B + D\right] \\
&\qquad = \left[-D^{-1}C(sI - E)^{-1}BD^{-1} + D^{-1}\right]\left[C(sI - A)^{-1}B + D\right] \\
&\qquad = -D^{-1}C(sI - E)^{-1}BD^{-1}C(sI - A)^{-1}B - D^{-1}C(sI - E)^{-1}B \\
&\qquad\qquad + D^{-1}C(sI - A)^{-1}B + I.
\end{aligned}$$

However, according to the hint

$$\begin{aligned}
&-D^{-1}C(sI - E)^{-1}BD^{-1}C(sI - A)^{-1}B \\
&\qquad = -D^{-1}C(sI - \underbrace{(A - BD^{-1}C)}_{X})^{-1}\underbrace{BD^{-1}C}_{-(X-Y)}(sI - \underbrace{A}_{Y})^{-1}B \\
&\qquad = D^{-1}C(sI - E)^{-1}B - D^{-1}C(sI - A)^{-1}B
\end{aligned}$$

and therefore

$$\begin{aligned}
&\left[G(sI - E)^{-1}F + H\right]\left[C(sI - A)^{-1}B + D\right] \\
&\qquad = D^{-1}C(sI - E)^{-1}B - D^{-1}C(sI - A)^{-1}B \\
&\qquad\qquad - D^{-1}C(sI - E)^{-1}B + D^{-1}C(sI - A)^{-1}B + I = I
\end{aligned}$$

and we are done.