EE363 Homework 3

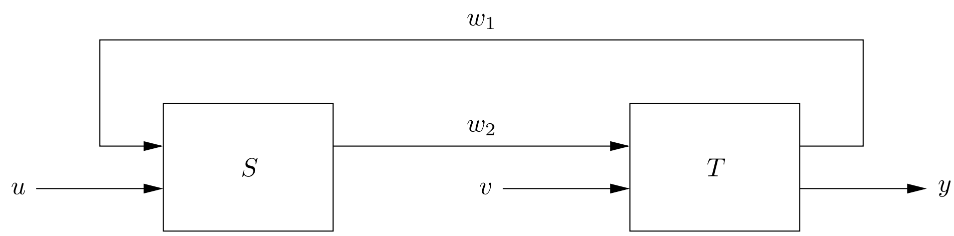
Spring 2026

10.1620. Volume preserving flows. Suppose we have a set $S \subseteq \mathbb{R}^n$ and a linear dynamical system $\dot{x} = Ax$. We can propagate S along the 'flow' induced by the linear dynamical system by considering

$$S(t) = e^{At}S = \{ \ e^{At}s \mid s \in S \ \}.$$

Thus, $S(t)$ is the image of the set S under the linear transformation e^{tA}. What are the conditions on A so that the flow preserves volume, *i.e.*, $\mathrm{vol}\, S(t) = \mathrm{vol}\, S$ for all t? Can the flow $\dot{x} = Ax$ be stable? *Hint:* if $F \in \mathbb{R}^{n \times n}$ then $\mathrm{vol}(FS) = |\det F| \,\mathrm{vol}\, S$, where $FS = \{ \ Fs \mid s \in S \ \}$.

13.1940. Interconnection of linear systems. Often a linear system is described in terms of a block diagram showing the interconnections between components or subsystems, which are themselves linear systems. In this problem you consider the specific interconnection shown below:



Here, there are two subsystems S and T. Subsystem S is characterized by

$$\dot{x} = Ax + B_1 u + B_2 w_1, \qquad w_2 = Cx + D_1 u + D_2 w_1,$$

and subsystem T is characterized by

$$\dot{z} = Fz + G_1 v + G_2 w_2, \qquad w_1 = H_1 z, \qquad y = H_2 z + J w_2.$$

We don't specify the dimensions of the signals (which can be vectors) or matrices here. You can assume all the matrices are the correct (*i.e.*, compatible) dimensions. Note that the subscripts in the matrices above, as in B_1 and B_2, refer to different matrices. Now the problem. Express the overall system as a single linear dynamical system with input, state, and output given by

$$\begin{bmatrix} u \\ v \end{bmatrix}, \qquad \begin{bmatrix} x \\ z \end{bmatrix}, \qquad y,$$

respectively. Be sure to explicitly give the input, dynamics, output, and feedthrough matrices of the overall system. If you need to make any assumptions about the rank or invertibility of any matrix you encounter in your derivations, go ahead. But be sure to let us know what assumptions you are making.

18.2840. Minimum energy required to steer the state to zero. Consider a controllable discrete-time system $x(t+1) = Ax(t) + Bu(t)$, $x(0) = x_0$. Let $E(x_0)$ denote the minimum energy required to drive the state to zero, *i.e.*

$$E(x_0) = \min \left\{ \sum_{\tau=0}^{t-1} \|u(\tau)\|^2 \mid x(t) = 0 \right\}.$$

An engineer argues as follows:

> This problem is like the minimum energy reachability problem, but 'turned backwards in time' since here we steer the state from a given state to zero, and in the reachability problem we steer the state from zero to a given state. The system $z(t+1) = A^{-1}z(t) - A^{-1}Bv(t)$ is the same as the given one, except time is running backwards. Therefore $E(x_0)$ is the same as the minimum energy required for z to reach x_0 (a formula for which can be found in the lecture notes).

Either justify or refute the engineer's statement. You can assume that A is invertible.

18.2870. Alternating input reachability. We consider a linear dynamical system with n states and 2 inputs,

$$x(t+1) = Ax(t) + Bu(t), \quad t = 0, 1, \ldots,$$

where $A \in \mathbb{R}^{n \times n}$, $B = [b_1\ b_2] \in \mathbb{R}^{n \times 2}$, $x(t) \in \mathbb{R}^n$ is the state, and $u(t) = (u_1(t), u_2(t)) \in \mathbb{R}^2$ is the input, at time t. We assume that $x(0) = 0$.

We say that an input sequence $u(0), u(1), \ldots$ is an *alternating* input sequence if $u_1(t) = 0$ for $t = 1, 3, 5, \ldots$ and $u_2(t) = 0$ for $t = 0, 2, 4, \ldots$, *i.e.*,

$$u(0) = \begin{bmatrix} u_1(0) \\ 0 \end{bmatrix}, \quad u(1) = \begin{bmatrix} 0 \\ u_2(1) \end{bmatrix}, \quad u(2) = \begin{bmatrix} u_1(2) \\ 0 \end{bmatrix}, \quad u(3) = \begin{bmatrix} 0 \\ u_2(3) \end{bmatrix}, \quad \ldots.$$

In contrast, we'll refer to an input sequence as a *standard* input sequence if both inputs can be nonzero at each time t.

We are given a target state $x_{\rm des} \in \mathbb{R}^n$, and a time horizon $N \geq n$.

a) Suppose we can find an alternating input sequence so that $x(2N) = x_{\rm des}$. Can we *always* find a standard input sequence so that $x(N) = x_{\rm des}$? In other words, if we can drive the state to $x_{\rm des}$ in $2N$ steps with an alternating input sequence, can we always find an input sequence that uses both inputs at each time step, and drives the state to $x_{\rm des}$ in N steps?

b) Is the converse true? Suppose we can find a standard input sequence so that $x(N) = x_{\rm des}$. Can we *always* find an alternating input sequence so that $x(2N) = x_{\rm des}$?

By *always*, we mean for any A, b_1, b_2, $x_{\rm des}$, and $N \geq n$. So, for example, if your answer is 'Yes' for part (a), you are saying that for any A, b_1, b_2, $x_{\rm des}$ and $N \geq n$, if we can find an alternating input sequence so that $x(2N) = x_{\rm des}$, then we can also find a standard input sequence so that $x(N) = x_{\rm des}$.

In your solution for parts (a) and (b) you should first state your answer, which must be either 'Yes' or 'No'. If your answer is 'Yes', you must provide a justification, and if your answer is ‘No’, you must provide a counterexample (and you must explain clearly why it is a counterexample). Your solution must be short; we won’t read more than one page. You may use any of the concepts from the class (*e.g.*, eigenvalues, pseudo-inverse, singular values, controllability, *etc.*).