EE363 Homework 3 Solutions

Spring 2026

10.1620. Volume preserving flows. Suppose we have a set $S \subseteq \mathbb{R}^n$ and a linear dynamical system $\dot{x} = Ax$. We can propagate S along the 'flow' induced by the linear dynamical system by considering

$$S(t) = e^{At}S = \{ e^{At}s \mid s \in S \}.$$

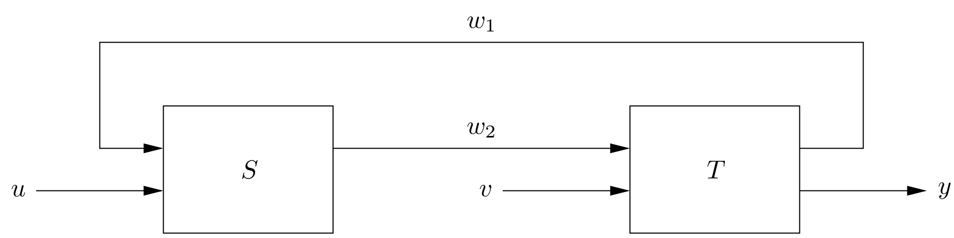
Thus, $S(t)$ is the image of the set S under the linear transformation e^{tA}. What are the conditions on A so that the flow preserves volume, *i.e.*, $\mathrm{vol}\, S(t) = \mathrm{vol}\, S$ for all t? Can the flow $\dot{x} = Ax$ be stable? *Hint:* if $F \in \mathbb{R}^{n \times n}$ then $\mathrm{vol}(FS) = |\det F| \,\mathrm{vol}\, S$, where $FS = \{ Fs \mid s \in S \}$.

Solution. The linear dynamical system $\dot{x} = Ax$ defines a linear transformation e^{tA} from $\mathbb{R}^n$ to $\mathbb{R}^n$. This linear transformation maps the set $S \subseteq \mathbb{R}^n$ to the set $S(t) \subseteq \mathbb{R}^n$ and therefore

$$\mathrm{vol}(S(t)) = \mathrm{vol}(e^{tA}S) = |\det e^{tA}| \,\mathrm{vol}\, S.$$

The flow preserves volume if and only if $\mathrm{vol}(S(t)) = \mathrm{vol}\, S$ or $|\det e^{tA}| = 1$. According to problem (**??**), $\det e^{tA} = e^{\mathrm{trace}\, At}$ and therefore the condition on A for the flow to preserve volume becomes $e^{\mathrm{trace}\, At} = 1$ or $\mathrm{trace}\, At = t\, \mathrm{trace}\, A = 0$ which implies that $\mathrm{trace}\, A = 0$. If the flow $\dot{x} = Ax$ preserves volume it cannot be stable. A volume preserving flow has $\mathrm{trace}\, A = 0$, or the sum of the eigenvalues of A should be zero. This implies that at least one eigenvalue has a nonnegative real part, so the system isn't stable. What we have shown in this problem is known as *Liouville's theorem.*

13.1940. Interconnection of linear systems. Often a linear system is described in terms of a block diagram showing the interconnections between components or subsystems, which are themselves linear systems. In this problem you consider the specific interconnection shown below:



Here, there are two subsystems S and T. Subsystem S is characterized by

$$\dot{x} = Ax + B_1u + B_2w_1, \qquad w_2 = Cx + D_1u + D_2w_1,$$

and subsystem T is characterized by

$$\dot{z} = Fz + G_1v + G_2w_2, \qquad w_1 = H_1z, \qquad y = H_2z + Jw_2.$$

We don't specify the dimensions of the signals (which can be vectors) or matrices here. You can assume all the matrices are the correct (*i.e.*, compatible) dimensions. Note that the subscripts in the matrices above, as in B_1 and B_2, refer to different matrices. Now the problem. Express the overall system as a single linear dynamical system with input, state, and output given by

$$\begin{bmatrix} u \\ v \end{bmatrix}, \qquad \begin{bmatrix} x \\ z \end{bmatrix}, \qquad y,$$

respectively. Be sure to explicitly give the input, dynamics, output, and feedthrough matrices of the overall system. If you need to make any assumptions about the rank or invertibility of any matrix you encounter in your derivations, go ahead. But be sure to let us know what assumptions you are making.

Solution. This one is easier than it might appear. All we need to do is write down all the equations for this system, and massage them to be in the form of a linear dynamical system with the given input, state, and output. Substituting the expression for w_1 into the first set of equations gives

$$\dot{x} = Ax + B_1 u + B_2 H_1 z, \qquad w_2 = Cx + D_1 u + D_2 H_1 z.$$

Similarly, substituting the expression for w_2 into the second set of equations yields

$$\dot{z} = Fz + G_1 v + G_2(Cx + D_1 u + D_2 H_1 z), \qquad y = H_2 z + J(Cx + D_1 u + D_2 H_1 z).$$

Now we just put this together into the required form:

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A & B_2 H_1 \\ G_2 C & F + G_2 D_2 H_1 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B_1 & 0 \\ G_2 D_1 & G_1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

and

$$y = \begin{bmatrix} JC & H_2 + J D_2 H_1 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} J D_1 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

18.2840. Minimum energy required to steer the state to zero. Consider a controllable discrete-time system $x(t+1) = Ax(t) + Bu(t)$, $x(0) = x_0$. Let $E(x_0)$ denote the minimum energy required to drive the state to zero, *i.e.*

$$E(x_0) = \min \left\{ \sum_{\tau=0}^{t-1} \|u(\tau)\|^2 \mid x(t) = 0 \right\}.$$

An engineer argues as follows:

> This problem is like the minimum energy reachability problem, but 'turned backwards in time' since here we steer the state from a given state to zero, and in the reachability problem we steer the state from zero to a given state. The system $z(t+1) = A^{-1} z(t) - A^{-1} B v(t)$ is the same as the given one, except time is running backwards. Therefore $E(x_0)$ is the same as the minimum energy required for z to reach x_0 (a formula for which can be found in the lecture notes).

Either justify or refute the engineer's statement. You can assume that A is invertible.

Solution. The backwards system is

$$z(t) = A^{-1}z(t+1) - A^{-1}Bv(t)$$

The energy required for the state z to reach x_0 is given by

$$\mathcal{E}_z = x_0^\mathsf{T}\left[\sum_{\tau=0}^{t_f-1} A^{-\tau-1}BB^\mathsf{T}(A^{-\tau-1})^\mathsf{T}\right]^{-1} x_0 = x_0^\mathsf{T}\left[\sum_{\tau=1}^{t_f} A^{-\tau}BB^\mathsf{T}(A^{-\tau})^\mathsf{T}\right]^{-1} x_0$$

and the energy required for the state x to go from x_0 to 0 is

$$\mathcal{E}_x = x_0^\mathsf{T}(A^{t_f})^\mathsf{T}\left[\sum_{\tau=0}^{t_f-1} A^{\tau}BB^\mathsf{T}(A^{\tau})^\mathsf{T}\right]^{-1} A^{t_f}x_0$$

Making the change of variables $t = t_f - \tau$ we can rewrite

$$\mathcal{E}_x = x_0^\mathsf{T}\left[\sum_{t=t_f}^{1} A^{-t}BB^\mathsf{T}(A^{-t})^\mathsf{T}\right]^{-1} x_0 = \mathcal{E}_z$$

and we conclude that the engineer's statement is correct.

18.2870. Alternating input reachability. We consider a linear dynamical system with n states and 2 inputs,

$$x(t+1) = Ax(t) + Bu(t), \quad t = 0, 1, \ldots,$$

where $A \in \mathbb{R}^{n\times n}$, $B = [b_1\ b_2] \in \mathbb{R}^{n\times 2}$, $x(t) \in \mathbb{R}^n$ is the state, and $u(t) = (u_1(t), u_2(t)) \in \mathbb{R}^2$ is the input, at time t. We assume that $x(0) = 0$.

We say that an input sequence $u(0), u(1), \ldots$ is an *alternating* input sequence if $u_1(t) = 0$ for $t = 1, 3, 5, \ldots$ and $u_2(t) = 0$ for $t = 0, 2, 4, \ldots$, *i.e.*,

$$u(0) = \begin{bmatrix} u_1(0) \\ 0 \end{bmatrix}, \quad u(1) = \begin{bmatrix} 0 \\ u_2(1) \end{bmatrix}, \quad u(2) = \begin{bmatrix} u_1(2) \\ 0 \end{bmatrix}, \quad u(3) = \begin{bmatrix} 0 \\ u_2(3) \end{bmatrix}, \quad \ldots.$$

In contrast, we'll refer to an input sequence as a *standard* input sequence if both inputs can be nonzero at each time t.

We are given a target state $x_{\rm des} \in \mathbb{R}^n$, and a time horizon $N \geq n$.

a) Suppose we can find an alternating input sequence so that $x(2N) = x_{\rm des}$. Can we *always* find a standard input sequence so that $x(N) = x_{\rm des}$? In other words, if we can drive the state to $x_{\rm des}$ in $2N$ steps with an alternating input sequence, can we always find an input sequence that uses both inputs at each time step, and drives the state to $x_{\rm des}$ in N steps?

b) Is the converse true? Suppose we can find a standard input sequence so that $x(N) = x_{\rm des}$. Can we *always* find an alternating input sequence so that $x(2N) = x_{\rm des}$?

By *always*, we mean for any A, b_1, b_2, x_{des}, and $N \geq n$. So, for example, if your answer is 'Yes' for part (a), you are saying that for any A, b_1, b_2, x_{des} and $N \geq n$, if we can find an alternating input sequence so that $x(2N) = x_{\text{des}}$, then we can also find a standard input sequence so that $x(N) = x_{\text{des}}$.

In your solution for parts (a) and (b) you should first state your answer, which must be either 'Yes' or 'No'. If your answer is 'Yes', you must provide a justification, and if your answer is 'No', you must provide a counterexample (and you must explain clearly why it is a counterexample). Your solution must be short; we won't read more than one page. You may use any of the concepts from the class (*e.g.*, eigenvalues, pseudo-inverse, singular values, controllability, *etc.*).

Solution. We have two statements:

I. We can find an alternating input sequence so that $x(2N) = x_{\text{des}}$.

II. We can find a standard input sequence so that $x(N) = x_{\text{des}}$.

Statement I is equivalent to $x_{\text{des}} \in \text{range}(\mathcal{C}_a)$, where

$$\mathcal{C}_a = \begin{bmatrix} b_2 & Ab_1 & A^2b_2 & A^3b_1 & \cdots & A^{2N-2}b_2 & A^{2N-1}b_1 \end{bmatrix}.$$

Similarly, statement II is equivalent to $x_{\text{des}} \in \text{range}(\mathcal{C}_s)$, where

$$\mathcal{C}_s = \begin{bmatrix} b_1 & b_2 & Ab_1 & Ab_2 & \cdots & A^{N-1}b_1 & A^{N-1}b_2 \end{bmatrix}.$$

a) The statement 'I implies II for any A, b_1, b_2, x_{des}, and $N \geq n$' is true. Here is our justification. The claim is true if and only if

$$\text{range}(\mathcal{C}_a) \subseteq \text{range}(\mathcal{C}_s),$$

for any A, b_1, b_2 and $N \geq n$. By the Cayley-Hamilton theorem, A^p can be written as a linear combination of $I, A, A^2, \ldots, A^{n-1}$, for any $p \in \mathbb{Z}_+$. This means that $A^p b_1$ can be written as a linear combination of $b_1, Ab_1, A^2b_1, \ldots, A^{n-1}b_1$, and similarly $A^p b_2$ can be written as a linear combination of $b_2, Ab_2, A^2b_2, \ldots, A^{n-1}b_2$. Thus the columns of the matrix $\mathcal{C}_a$ can be written as linear combinations of the columns of the matrix $\mathcal{C}_s$, which shows that $\text{range}(\mathcal{C}_a) \subseteq \text{range}(\mathcal{C}_s)$ for any A, b_1, b_2, and $N \geq n$.

b) The statement that 'II implies I for any A, b_1, b_2, x_{des}, and $N \geq n$' is not true. To see this, suppose b_1 and b_2 are independent, and $A = 0$. In this case, $\text{range}(\mathcal{C}_a) = \text{span}\{b_2\}$, but $\text{range}(\mathcal{C}_s) = \text{span}\{b_1, b_2\}$. Let $x_{\text{des}} = b_1$. Clearly, $x_{\text{des}} \in \text{range}(\mathcal{C}_s)$, which means that we can find a standard input sequence that gives $x(N) = x_{\text{des}}$. However, $x_{\text{des}} \notin \text{range}(\mathcal{C}_a)$, and so we cannot find an alternating input sequence that gives $x(2N) = x_{\text{des}}$.

To be completely concrete, we can take

$$A = 0 \in \mathbb{R}^{2\times 2}, \quad b_1 = e_1, \quad b_2 = e_2, \quad N = 2, \quad x_{\text{des}} = b_1.$$

For this example there is a standard input sequence that achieves $x(N) = x_{\text{des}}$, but there is no alternating input sequence that achieves $x(2N) = x_{\text{des}}$.