EE363 Homework 4

Spring 2026

10.1600. Positive quadrant invariance. We consider a system $\dot{x} = Ax$ with $x(t) \in \mathbb{R}^2$ (although the results of this problem can be generalized to systems of higher dimension). We say the system is *positive quadrant invariant* (PQI) if whenever $x_1(T) \geq 0$ and $x_2(T) \geq 0$, we have $x_1(t) \geq 0$ and $x_2(t) \geq 0$ for all $t \geq T$. In other words, if the state starts inside (or enters) the positive (*i.e.*, first) quadrant, then the state remains indefinitely in the positive quadrant.

a) Find the precise conditions on A under which the system $\dot{x} = Ax$ is PQI. Try to express the conditions in the simplest form.

b) *True or False:* if $\dot{x} = Ax$ is PQI, then the eigenvalues of A are real.

15.2420. Energy storage efficiency in a linear dynamical system. We consider the discrete-time linear dynamic system

$$x(t+1) = Ax(t) + Bu(t), \quad y(t) = Cx(t),$$

where $x(t) \in \mathbb{R}^n$, and $u(t), y(t) \in \mathbb{R}$. The initial state is zero, *i.e.*, $x(0) = 0$. We apply an input sequence $u(0), \dots, u(N-1)$, and are interested in the output over the next N samples, *i.e.*, $y(N), \dots, y(2N-1)$. (We take $u(t) = 0$ for $t \geq N$.) We define the *input energy* as

$$\mathcal{E}_{\text{in}} = \sum_{t=0}^{N-1} u(t)^2,$$

and similarly, the output energy is defined as

$$\mathcal{E}_{\text{out}} = \sum_{t=N}^{2N-1} y(t)^2.$$

How would you choose the (nonzero) input sequence $u(0), \dots, u(N-1)$ to maximize the ratio of output energy to input energy, *i.e.*, to maximize $\mathcal{E}_{\text{out}}/\mathcal{E}_{\text{in}}$? What is the maximum value the ratio $\mathcal{E}_{\text{out}}/\mathcal{E}_{\text{in}}$ can have?

19.2200. Observability of a discretized system. Consider the continuous-time system $\dot{x} = Ax$, $y = Cx$ where

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix} \; A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

This is the harmonic oscillator system from the lecture.

a) Verify that the system is observable from the observability matrix $\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix}$. You may use any programming language you like, or compute it by hand.

b) Express $y(t)$ in terms of A, C, the initial state x_0 and any standard matrix operations you find useful. If you were to measure the continuous-time signal $y(t)$ over some interval $t \in [0, T]$, could you determine a unique x_0? Why or why not?

c) Suppose we discretize this system. Write down the discretized matrices A_d and C_d in terms of A, C, any matrix operations you find useful, and the sampling period Δt.

d) Is the discretized system observable for all possible values of Δt? If not, what value(s) of Δt makes the system unobservable and why? You can provide an intuitive explanation using the concepts we have learned so far; a formal proof is not necessary.