EE363 Homework 4

Spring 2026

10.1600. Positive quadrant invariance. We consider a system $\dot{x} = Ax$ with $x(t) \in \mathbb{R}^2$ (although the results of this problem can be generalized to systems of higher dimension). We say the system is *positive quadrant invariant* (PQI) if whenever $x_1(T) \geq 0$ and $x_2(T) \geq 0$, we have $x_1(t) \geq 0$ and $x_2(t) \geq 0$ for all $t \geq T$. In other words, if the state starts inside (or enters) the positive (*i.e.*, first) quadrant, then the state remains indefinitely in the positive quadrant.

a) Find the precise conditions on A under which the system $\dot{x} = Ax$ is PQI. Try to express the conditions in the simplest form.

b) *True or False:* if $\dot{x} = Ax$ is PQI, then the eigenvalues of A are real.

Solution.

a) This problem can be solved by several methods. The simplest method is probably this: at the quadrant boundaries, the derivative $\dot{x} = Ax$ must point *into* the quadrant (or at least, along the quadrant boundary). Therefore, if n is the *inward* normal vector to the boundary at coordinates x we must have $n^\mathsf{T} Ax \geq 0$. The first boundary is characterized by $x_1 = 0$ and $x_2 \geq 0$. The inward normal vector at all points of this boundary is simply $n = e_1 = [1\ 0]^\mathsf{T}$. Therefore we should have $e_1^\mathsf{T} Ax \geq 0$. Similarly, for the other boundary characterized by $x_1 \geq 0$ and $x_2 = 0$ we require $e_2^\mathsf{T} Ax \geq 0$ If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then the conditions become

$$\begin{aligned} x_1 = 0, \quad x_2 \geq 0, \quad e_1^\mathsf{T} Ax = a_{11}x_1 + a_{12}x_2 = a_{12}x_2 \geq 0 \quad &\Longrightarrow \quad a_{12} \geq 0, \\ x_2 = 0, \quad x_1 \geq 0, \quad e_2^\mathsf{T} Ax = a_{21}x_2 + a_{22}x_2 = a_{21}x_1 \geq 0 \quad &\Longrightarrow \quad a_{21} \geq 0. \end{aligned}$$

In other words, the off-diagonal elements must be non-negative. In fact, systems $\dot{x} = Ax$ of *any* degree are *positive orthant invariant* (POI) if and only if all off-diagonal elements of A are non-negative. The problem can also be solved by studying whether all entries of the matrix e^{At} are nonnegative. That is quite a bit more complicated, but ends with the same condition.

b) *True*. The characteristic equation of the system is

$$\det(sI - A) = \det \begin{bmatrix} s - a_{11} & -a_{12} \\ -a_{21} & s - a_{22} \end{bmatrix} = s^2 - (a_{11} + a_{22})s + a_{11}a_{22} - a_{12}a_{21}.$$

In order for the eigenvalues to be complex, the discriminant of the quadratic equation, $b^2 - 4ac$, must be negative. But

$$(a_{11} + a_{22})^2 - 4a_{11}a_{22} + 4a_{12}a_{21} = (a_{11} - a_{22})^2 + 4a_{12}a_{21} \geq 0.$$

So, the eigenvalues must indeed be real. We can also derive this result another way. If the system has complex eigenvalues, then x_1 has the form $x_1(t) = ae^{\sigma t}\cos(\omega t + \phi)$ (and similarly for x_2). Obviously then there are positive times when $x_1(t) < 0$, contradicting PQI.

15.2420. Energy storage efficiency in a linear dynamical system. We consider the discrete-time linear dynamic system

$$x(t+1) = Ax(t) + Bu(t), \quad y(t) = Cx(t),$$

where $x(t) \in \mathbb{R}^n$, and $u(t), y(t) \in \mathbb{R}$. The initial state is zero, *i.e.*, $x(0) = 0$. We apply an input sequence $u(0), \ldots, u(N-1)$, and are interested in the output over the next N samples, *i.e.*, $y(N), \ldots, y(2N-1)$. (We take $u(t) = 0$ for $t \geq N$.) We define the *input energy* as

$$\mathcal{E}_{\text{in}} = \sum_{t=0}^{N-1} u(t)^2,$$

and similarly, the output energy is defined as

$$\mathcal{E}_{\text{out}} = \sum_{t=N}^{2N-1} y(t)^2.$$

How would you choose the (nonzero) input sequence $u(0), \ldots, u(N-1)$ to maximize the ratio of output energy to input energy, *i.e.*, to maximize $\mathcal{E}_{\text{out}}/\mathcal{E}_{\text{in}}$? What is the maximum value the ratio $\mathcal{E}_{\text{out}}/\mathcal{E}_{\text{in}}$ can have?

Solution. For the discrete time system $x(t+1) = Ax(t) + Bu(t)$, $y(t) = Cx(t)$, we can write the state at time t in terms of the past inputs as

$$x(t) = \begin{bmatrix} B & AB & \ldots & A^{t-1}B \end{bmatrix} \begin{bmatrix} u(t-1) \\ \vdots \\ u(0) \end{bmatrix},$$

and the output $y(t)$ as

$$y(t) = \begin{bmatrix} CB & CAB & \ldots & CA^{t-1}B \end{bmatrix} \begin{bmatrix} u(t-1) \\ \vdots \\ u(0) \end{bmatrix}.$$

Since the input is zero from time $t = N$ onwards, the state is propagated forward by A at each sample, *i.e.*, $x(t+1) = Ax(t)$, and so for $t \geq N$, we can write the output in terms of the input from $t = 0$ to $t = N-1$ as

$$y(t) = \begin{bmatrix} CA^{t-N}B & CA^{t-N+1}B & \ldots & CA^{t-1}B \end{bmatrix} \begin{bmatrix} u(N-1) \\ \vdots \\ u(0) \end{bmatrix}.$$

Stacking up the $y(t)$s, $t = N, \ldots, 2N-1$, we obtain the equation

$$\begin{bmatrix} y(N) \\ \vdots \\ \\ y(2N-1) \end{bmatrix} = \begin{bmatrix} CB & CAB & \ldots & CA^{N-1}B \\ CAB & CA^2B & \ldots & CA^NB \\ \vdots & \vdots & & \vdots \\ CA^{N-1}B & CA^NB & \ldots & CA^{2N-2}B \end{bmatrix} \begin{bmatrix} u(N-1) \\ \vdots \\ \\ u(0) \end{bmatrix},$$

or $Y = \mathcal{C}U$; and we are interested in maximizing $\|Y\|/\|U\|$, subject to $\|U\| \leq 1$. But this is an SVD problem- the maximum value of $\|Y\|/\|U\|$ is equal to the largest singular value of $\mathcal{C}$, and the sequence of inputs u is precisely the right singular vector corresponding to $\sigma_{\max}(\mathcal{C})$!

19.2200. Observability of a discretized system. Consider the continuous-time system $\dot{x} = Ax$, $y = Cx$ where

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix} \;\; A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

This is the harmonic oscillator system from the lecture.

a) Verify that the system is observable from the observability matrix $\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix}$. You may use any programming language you like, or compute it by hand.

b) Express $y(t)$ in terms of A, C, the initial state x_0 and any standard matrix operations you find useful. If you were to measure the continuous-time signal $y(t)$ over some interval $t \in [0, T]$, could you determine a unique x_0? Why or why not?

c) Suppose we discretize this system. Write down the discretized matrices A_d and C_d in terms of A, C, any matrix operations you find useful, and the sampling period Δt.

d) Is the discretized system observable for all possible values of Δt? If not, what value(s) of Δt makes the system unobservable and why? You can provide an intuitive explanation using the concepts we have learned so far; a formal proof is not necessary.

Solution.

a) The observability matrix works out to be $\mathcal{O} = I_2$ (the 2x2 identity matrix). $\mathcal{O}$ is full rank so the system is observable.

b) Since the system is observable, we can uniquely identify x_0.

c) The discretized matrices are $C_d = C$ and $A_d = \exp(A\Delta t)$ where exp is the matrix exponential.

d) Intuitively, one example of an unobservable mode is if our system oscillates at *exactly* the same frequency we sample at. Thus, the value $\Delta t = k\pi$ makes the system unobservable. Here is a short proof.

   From the slides, we know that

$$\begin{aligned} \exp \Delta t A &= I + \Delta t A + \frac{(\Delta t)^2 A^2}{2} + \frac{(\Delta t)^3 A^3}{3!} + \ldots \\ &= (\cos(\Delta t) I + \sin(\Delta t) A. \end{aligned}$$

   Let $\mathcal{O}_d = \begin{bmatrix} C_d \\ C_d A_d \end{bmatrix}$ be the discretized observability matrix. To produce a non-full-rank $\mathcal{O}_d$, we must have $C_d = \alpha C_d A_d$ for some nonzero $\alpha \in \mathbb{R}$. For example, $A_d = \pm I$. Since $A_d = (\cos(\Delta t) I + \sin(\Delta t) A$, we pick $\Delta t = k\pi$ to satisfy this condition.