EE363 Homework 5

Spring 2026

22.1000. Controlling a mixing tank. Consider a mixing tank with cold and hot water supplies at constant supply temperatures T_C and T_H, respectively. Let q_C and q_H denote the input flow rate for each supply and treat them as control inputs. Let h and T be the height and temperature of the water in the tank; these are our state variables. The differential equations governing this system are

$$\begin{aligned}\dot{h} &= \frac{1}{\alpha}\left(q_C + q_H - \beta\sqrt{h}\right)\\ \dot{T} &= \frac{1}{\alpha h}\left(q_C(T_C - T) + q_H(T_H - T)\right)\end{aligned}$$

where α is the area of the tank and the term $\beta\sqrt{h}$, where β is a constant, dictates the rate at which water is drained. Using the standard state and input notation, let $x_1 = h, x_2 = T, u_1 = q_C, u_2 = q_H$.

a) Suppose we want the steady-state values to be $(h^{\text{eq}}, T^{\text{eq}})$, where $h^{\text{eq}} > 0$ and $T^{\text{eq}} \in [T_C, T_H]$. What are the corresponding steady-state input values, u_1^{eq}, u_2^{eq}?

b) We can linearize this system around the equilibrium $(h^{eq}, T^{eq}, u_1^{eq}, u_2^{eq})$ using standard techniques:

$$A = \left.\frac{\partial f}{\partial x}\right|_{eq}, \quad B = \left.\frac{\partial f}{\partial u}\right|_{eq}$$

The linearized matrices are

$$A = \begin{bmatrix} -\dfrac{\beta}{2\alpha\sqrt{h^{eq}}} & 0 \\ 0 & -\dfrac{\beta}{\alpha\sqrt{h^{eq}}} \end{bmatrix}, \quad B = \frac{1}{\alpha}\begin{bmatrix} 1 & 1 \\ \dfrac{T_C - T^{eq}}{h^{eq}} & \dfrac{T_H - T^{eq}}{h^{eq}} \end{bmatrix}$$

Note that A is diagonal, which reflects that height and temperature dynamics decouple at steady-state.

Suppose $T_C = 10^\circ, T_H = 90^\circ, \alpha = 3\text{m}^2, \beta = 0.155\text{m}^{5/2}/\text{s}$, and $h^\star = 1\text{m}, T^\star = 25^\circ$. Evaluate A and B in part (b) for these values. Report the eigenvalues of A (you should do this numerically in your favorite programming language). Is the system (A, B) controllable at these values?

c) Design a controller of the form $\tilde{u} = L\tilde{x}$ for the values in part (c) using the pole placement method. Compute feedback gains L using your favorite programming language for the following closed-loop eigenvalue locations:

$$\{-0.1, -0.2\}, \quad \{-0.3, -0.6\}, \quad \{-1, -2\}.$$

Hint: in Python you can use `scipy.signal.place_poles`.

d) Optional: simulate the nonlinear model for each controller, You may observe some non-physical behavior (such as $h(t) \leq 0$ or negative flow rates. Why do you think this occurs?

22.1200. Deadbeat control. Consider the discrete-time system

$$x_{k+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 0//1 \end{bmatrix} u_k.$$

Define $A_{\text{cl}} = A + BL$.

A state feedback law that places all eigenvalues of of a discrete-time system at zero is called a deadbeat controller. For an n^{th} order system, this ensures $A_{\text{cl}}^n = 0$ thus, closed-loop trajectories converge to zero in finite time.

Finite-time convergence can't be achieved in continuous time with linear state feedback because the matrix exponential $\exp A_{\text{cl}} t$ is never exactly zero at any finite time. Instead, trajectories approach the origin asymptotically. (An important caveat is that deadbeat control requires large feedback gains, increasing sensitivity to modeling errors and noise.)

a) Design a state feedback control $u_k = Lx_k$ that places both closed-loop eigenvalues at zero.

b) Show that $A_{\text{cl}}^2 = 0$. What does this imply for the closed-loop trajectories?

23.1000. Linear quadratic state tracking. We consider the system $x_{t+1} = Ax_t + Bu_t$. In the conventional LQR problem the goal is to make both the state and the input small. In this problem we study a generalization in which we want the state to follow a desired (possibly nonzero) trajectory as closely as possible. To do this we penalize the *deviations* of the state from the desired trajectory, *i.e.*, $x_t - x_t^d$, using the following cost function:

$$J = \sum_{\tau=0}^{N} \left(x_\tau - x_\tau^d\right)^\top Q \left(x_\tau - x_\tau^d\right) + \sum_{\tau=0}^{N-1} u_\tau^\top R u_\tau,$$

where we assume $Q = Q^\top \succeq 0$ and $R = R^\top \succ 0$. (The desired trajectory x_τ^d is given.) Compared with the standard LQR obejctive, we have an extra linear term (in x) and a constant term.

In this problem you will use dynamic programming to show that the cost-to-go function $V_t(z)$ for this problem has the form

$$z^\top P_t z + 2q_t^\top z + r_t,$$

with $P_t = P_t^\top \succeq 0$. (*i.e.*, it has quadratic, linear, and constant terms.)

a) Show that $V_n(z)$ has the given form.

b) Assuming $V_{t+1}(z)$ has the given form, show that the optimal input at time t can be written as

$$u_t^\star = K_t x_t + g_t,$$

where

$$K_t = -\left(R + B^\top P_{t+1} B\right)^{-1} B^\top P_{t+1} A, \quad g_t = -\left(R + B^\top P_{t_1} B\right)^{-1} B^\top q_{t+1}.$$

In other words, $u_t^\star$ is an affine (linear plus constant) function of the state x_t.

c) Use backward induction to show that $V_0(z), \ldots, V_N(z)$ all have the given form. Verify that

$$\begin{aligned}
P_t &= Q + A^\top P_{t+1} A - A^\top P_{t+1} B \left(R + B^\top P_{t+1} B\right)^{-1} B^\top P_{t+1} A, \\
q_t &= (A + BK_t)^\top q_{t+1} - Q x_t^d, \\
r_t &= r_{t+1} + x_t^d Q x_t^d + q_{t+1}^\top B g_t,
\end{aligned}$$

for $t = 0, \ldots, N-1$.

23.1100. When does a finite-horizon LQR problem have a time-invariant optimal state feedback gain?. Consider a discrete-time LQR problem with horizon $t = N$, with optimal input $u(t) = K_t x(t)$. Is there a choice of Q_f (that is symmetric and positive semidefinite) for which K_t is constant, *i.e.*, $K_0 = \cdots = K_{N-1}$?