EE363 Homework 6

Spring 2026

19.2881. Sensor selection and observer design. Consider the system $\dot{x} = Ax$, $y = Cx$, with

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(This problem concerns observer design so we've simplified things by not even including an input.) We consider observers that (exactly and instantaneously) reconstruct the state from the output and its derivatives. Such observers have the form

$$x(t) = F_0 y(t) + F_1 \frac{dy}{dt}(t) + \cdots + F_k \frac{d^k y}{dt^k}(t),$$

where $F_0, \ldots, F_k$ are matrices that specify the observer. (Of course we require this formula to hold for any trajectory of the system and any t, *i.e.*, the observer has to work!) Consider an observer defined by $F_0, \ldots, F_k$. We say the *degree* of the observer is the largest j such that $F_j \neq 0$. The degree gives the highest derivative of y used to reconstruct the state. If the ith columns of $F_0, \ldots, F_k$ are all zero, then the observer doesn't use the ith sensor signal $y_i(t)$ to reconstruct the state. We say the observer *uses* or *requires* the sensor i if at least one of the ith columns of $F_0, \ldots, F_k$ is nonzero.

a) What is the minimum number of sensors required for such an observer? List all combinations (*i.e.*, sets) of sensors, of this minimum number, for which there is an observer using only these sensors.

b) What is the minimum degree observer? List all combinations of sensors for which an observer of this minimum degree can be found.

22.1300. Stabilizing the magnetically-suspended ball. The magnetically suspended ball is modeled by

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = 9.8 - \frac{1}{15}\left(\frac{u}{x_1}\right)^2, \qquad y = x_1,$$

where x_1 and x_2 are the position and velocity of the ball, respectively, and u is the current.

Suppose we want to design the input u to stabilize the ball at $y^{\text{eq}} = 0.1$, *i.e.*, at the equilibrium point

$$x^{\text{eq}} = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}, \qquad u^{\text{eq}} = 1.212.$$

The linearized model around this equilibrium is

$$\dot{\tilde{x}} = \begin{bmatrix} 0 & 1 \\ 195.9 & 0 \end{bmatrix} \tilde{x} + \begin{bmatrix} 0 \\ -16.1 \end{bmatrix} \tilde{u}, \qquad \tilde{y} = \begin{bmatrix} 1 & 0 \end{bmatrix} \tilde{x},$$

where

$$\tilde{x} = x - x^{\text{eq}} = \begin{bmatrix} x_1 - x_1^{\text{eq}} \\ x_2 - x_2^{\text{eq}} \end{bmatrix}, \qquad \tilde{y} = y - y^{\text{eq}}, \qquad \tilde{u} = u - u^{\text{eq}}.$$

(a) Show that the linearized system is controllable.

(b) Design a state feedback controller $\tilde{u} = K\tilde{x}$ that asymptotically stabilizes the linearized system.

(c) *Optional:* Simulate the nonlinear system model with the controller designed in part (b), applied as $u = u^{\text{eq}} + \tilde{u}$. Select at least three different initial conditions, starting close to the equilibrium point and progressively going farther. Plot the outputs y.

23.1300. LQR with exponential weighting. A common variation on the LQR problem includes explicit time-varying weighting factors on the state and input costs,

$$J = \sum_{\tau=0}^{N-1} \gamma^\tau \left(x_\tau^T Q x_\tau + u_\tau^T R u_\tau\right) + \gamma^N x_N^T Q_f x_N,$$

where

$$x_{t+1} = Ax_t + Bu_t,$$

x_0 is given, and, as usual, $Q = Q^T \succeq 0$, $Q_f = Q_f^T \succeq 0$, and $R = R^T \succ 0$ are constant. The parameter γ, called the exponential weighting factor, is positive. For $\gamma = 1$, this reduces to the standard LQR cost function. For $\gamma < 1$, the penalty for future state and input deviations is smaller than in the present; in this case we call γ the discount factor or forgetting factor. When $\gamma > 1$, future costs are accentuated compared to present costs. This gives added incentive for the input to steer the state towards zero quickly.

(a) Note that we can find the input sequence $u_0^\star, \ldots, u_{N-1}^\star$ that minimizes J using standard LQR methods, by considering the state and input costs as time varying, with

$$Q_t = \gamma^t Q, \qquad R_t = \gamma^t R,$$

and final cost matrix $\gamma^N Q_f$. Thus, we know at least one way to solve the exponentially weighted LQR problem. Use this method to find the recursive equations that give $u^\star$.

(b) Exponential weights can also be incorporated directly into a dynamic programming formulation. We define

$$W_t(z) = \min \sum_{\tau=t}^{N-1} \gamma^{\tau-t} \left(x_\tau^T Q x_\tau + u_\tau^T R u_\tau\right) + \gamma^{N-t} x_N^T Q_f x_N,$$

where $x_t = z$, $x_{\tau+1} = Ax_\tau + Bu_\tau$, and the minimum is over $u_t, \ldots, u_{N-1}$. This is the minimum cost-to-go, if we started in state z at time t, with the time weighting also starting at t. Argue that we have

$$W_N(z) = z^T Q_f z,$$

$$W_t(z) = \min_w \left(z^T Q z + w^T R w + \gamma W_{t+1}(Az + Bw)\right),$$

and that the minimizing w is in fact $u_t^\star$. In other words, work out a backwards recursion for W_t, and give an expression for $u_t^\star$ in terms of W_t. Show that this method yields the same $u^\star$ as the first method.

(c) Yet another method can be used to find $u^\star$. Define a new system as

$$y_{t+1} = \gamma^{1/2} A y_t + \gamma^{1/2} B z_t, \qquad y_0 = x_0.$$

Argue that we have

$$y_t = \gamma^{t/2} x_t,$$

provided

$$z_t = \gamma^{t/2} u_t,$$

for $t = 0, \ldots, N-1$. With this choice of z, the exponentially weighted LQR cost J for the original system is given by

$$J = \sum_{\tau=0}^{N-1} \left(y_\tau^T Q y_\tau + z_\tau^T R z_\tau \right) + y_N^T Q_f y_N,$$

i.e., the unweighted LQR cost for the modified system. We can use the standard formulas to obtain the optimal input for the modified system $z^\star$, and from this, we can get $u^\star$. Do this, and verify that once again, you get the same $u^\star$.

23.1400. Discrete-time finite-horizon LQR. We consider the scalar discrete-time system

$$x_{t+1} = a x_t + u_t,$$

where x_0 is given. Consider the finite-horizon LQR problem

$$V_0(x_0) = \min_{u_0, u_1} \sum_{t=0}^{1} \left(x_t^2 + r u_t^2 \right) + s x_2^2,$$

where $r > 0$ and $s \geq 0$. Compute the optimal control inputs u_0 and u_1 as functions of the given initial state x_0, in terms of a, r, and s.