Solution via matrix exponential

- matrix exponential
- solving $\dot{x} = Ax$ via matrix exponential
- state transition matrix

Matrix exponential

define **matrix exponential** as

$$e^M = I + M + \frac{M^2}{2!} + \cdots$$

- converges for all $M \in \mathbb{R}^{n \times n}$
- looks like ordinary power series

  $$e^a = 1 + a + \frac{a^2}{2!} + \cdots$$

  with square matrices instead of scalars ...

Properties of the Matrix exponential

- $e^0 = I$ where $0 \in \mathbb{R}^{n \times n}$
- $e^{(M^\top)} = (e^M)^\top$
- $\dfrac{d}{dt} e^{tA} = A e^{tA} = e^{tA} A$

these follow from the definition

Matrix exponential solution of autonomous LDS

solution of $\dot{x} = Ax$, with $A \in \mathbb{R}^{n \times n}$ and constant, is

$$x(t) = e^{tA}x(0)$$

the matrix e^{tA} is called the *state transition matrix*, usually written $\Phi(t)$

generalizes scalar case: solution of $\dot{x} = ax$, with $a \in \mathbb{R}$ and constant, is

$$x(t) = e^{ta}x(0)$$

Properties of matrix exponential

- matrix exponential is *meant* to look like scalar exponential
- some things you'd guess hold for the matrix exponential (by analogy with the scalar exponential) do in fact hold
- but **many things you'd guess are wrong**

example: you might guess that $e^{A+B} = e^A e^B$, but it's false (in general)

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$e^A = \begin{bmatrix} 0.54 & 0.84 \\ -0.84 & 0.54 \end{bmatrix}, \qquad e^B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$e^{A+B} = \begin{bmatrix} 0.16 & 1.40 \\ -0.70 & 0.16 \end{bmatrix} \neq e^A e^B = \begin{bmatrix} 0.54 & 1.38 \\ -0.84 & -0.30 \end{bmatrix}$$

Matrix exponential

- if A and B commute, than $e^{A+B} = e^A e^B$
- because if $AB = BA$, then $e^{At}B = Be^{At}$, so

$$\begin{aligned}\frac{d}{dt}e^{At}e^{Bt} &= Ae^{At}e^{Bt} + e^{At}Be^{Bt} \\ &= (A+B)e^{At}e^{Bt}\end{aligned}$$

- hence $e^{At}e^{Bt} = e^{(A+B)t}$

Properties of matrix exponential

$$e^{A+B} = e^A e^B \text{ if } AB = BA$$

$i.e.$, product rule holds when A and B commute

thus for $t,\ s \in \mathbb{R}$, $e^{(tA+sA)} = e^{tA} e^{sA}$

with $s = -t$ we get

$$e^{tA} e^{-tA} = e^{tA-tA} = e^0 = I$$

so e^{tA} is nonsingular, with inverse

$$\left(e^{tA}\right)^{-1} = e^{-tA}$$

Example: matrix exponential

let's find e^{tA}, where $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

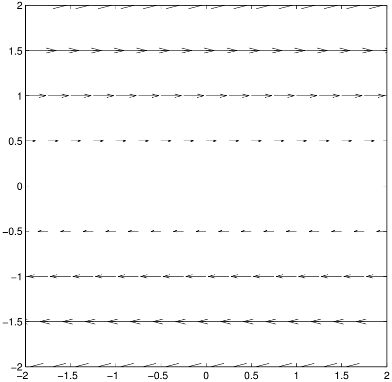
the power series gives

$$\begin{aligned} e^{tA} &= I + tA + \frac{t^2A^2}{2} + \frac{t^3A^3}{3!} + \cdots \\ &= I + tA \qquad \text{since } A^2 = 0 \\ &= \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \end{aligned}$$

we have $x(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} x(0)$

Example: Double integrator

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x$$



Example: Harmonic oscillator

$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, so state transition matrix is

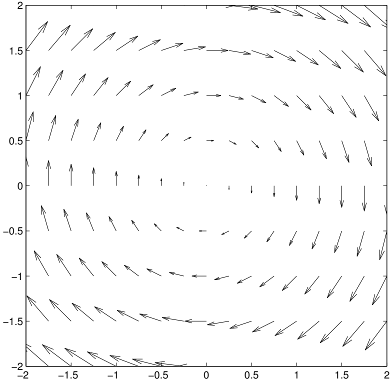
$$\begin{aligned} e^{tA} &= I + tA + \frac{t^2A^2}{2} + \frac{t^3A^3}{3!} + \frac{t^4A^4}{4!} \cdots \\ &= \left(1 - \frac{t^2}{2} + \frac{t^4}{4!} - \cdots\right) I + \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots\right) A \\ &= (\cos t)I + (\sin t)A \\ &= \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \end{aligned}$$

a rotation matrix ($-t$ radians)

so we have $x(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} x(0)$

Example: Harmonic oscillator

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x$$



Time transfer property

for $\dot{x} = Ax$ we know

$$x(t) = \Phi(t)x(0) = e^{tA}x(0)$$

interpretation: the matrix e^{tA} propagates initial condition into state at time t

more generally we have, for *any* t and τ,

$$x(\tau + t) = e^{tA}x(\tau)$$

(to see this, apply result above to $z(t) = x(t + \tau)$)

interpretation: the matrix e^{tA} propagates state t seconds forward in time (backward if $t < 0$)

Numerical integration of continuous system

compute approximate solution of $\dot{x} = Ax$, $x(0) = x_0$

suppose h is small time step (x doesn't change much in h seconds)

simple ('forward Euler') approximation:

$$x(t+h) \approx x(t) + h\dot{x}(t) = (I + hA)x(t)$$

by carrying out this recursion (discrete-time LDS), starting at $x(0) = x_0$, we get approximation

$$x(kh) \approx (I + hA)^k x(0)$$

(forward Euler is never used in practice)

Time transfer property

- forward Euler *approximate* state update, for small t:

$$x(\tau + t) \approx x(\tau) + t\dot{x}(\tau) = (I + tA)x(\tau)$$

- *exact* solution is

$$x(\tau + t) = e^{tA}x(\tau) = (I + tA + (tA)^2/2! + \cdots)x(\tau)$$

- forward Euler is just first two terms in series

Sampling a continuous-time system

suppose $\dot{x} = Ax$

sample x at times $t_1 \leq t_2 \leq \cdots$: define $z(k) = x(t_k)$

then $z(k+1) = e^{(t_{k+1} - t_k)A} z(k)$

for uniform sampling $t_{k+1} - t_k = h$, so

$$z(k+1) = e^{hA} z(k),$$

a discrete-time LDS (called *discretized version* of continuous-time system)

Piecewise constant system

consider *time-varying* LDS $\dot{x} = A(t)x$, with

$$A(t) = \begin{cases} A_0 & 0 \le t < t_1 \\ A_1 & t_1 \le t < t_2 \\ \vdots & \end{cases}$$

where $0 < t_1 < t_2 < \cdots$ (sometimes called jump linear system)

for $t \in [t_i, t_{i+1}]$ we have

$$x(t) = e^{(t-t_i)A_i} \cdots e^{(t_3-t_2)A_2} e^{(t_2-t_1)A_1} e^{t_1 A_0} x(0)$$

(matrix on righthand side is called state transition matrix for system, and denoted $\Phi(t)$)

Laplace transform of matrix valued function

suppose $z : \mathbb{R}_+ \to \mathbb{R}^{p\times q}$

Laplace transform: $\hat{z} = \mathcal{L}(z)$, where $\hat{z} : D \subseteq \mathbb{C} \to \mathbb{C}^{p\times q}$ is defined by

$$\hat{z}(s) = \int_0^\infty e^{-st} z(t)\, dt$$

- integral of matrix is done term-by-term
- convention: upper case denotes Laplace transform
- D is the *domain* or *region of convergence* of $\hat{z}$
- D includes at least $\{s \mid \Re s > a\}$, where a satisfies $|z_{ij}(t)| \le \alpha e^{at}$ for $t \ge 0$, $i = 1, \ldots, p$, $j = 1, \ldots, q$

Derivative property

$\mathcal{L}(\dot{z}) = s\hat{z}(s) - z(0)$

to derive, integrate by parts:

$$\begin{aligned} \mathcal{L}(\dot{z})(s) &= \int_0^\infty e^{-st}\dot{z}(t)\,dt \\ &= e^{-st}z(t)\Big|_{t=0}^{t\to\infty} + s\int_0^\infty e^{-st}z(t)\,dt \\ &= s\hat{z}(s) - z(0) \end{aligned}$$

Laplace transform solution of $\dot{x} = Ax$

consider continuous-time time-invariant (TI) LDS

$$\dot{x} = Ax$$

for $t \geq 0$, where $x(t) \in \mathbb{R}^n$

- take Laplace transform: $s\hat{x}(s) - x(0) = A\hat{x}(s)$
- rewrite as $(sI - A)\hat{x}(s) = x(0)$
- hence $\hat{x}(s) = (sI - A)^{-1}x(0)$
- take inverse transform

$$x(t) = \mathcal{L}^{-1}\left((sI - A)^{-1}\right)x(0)$$

Resolvent and state transition matrix

- $(sI - A)^{-1}$ is called the *resolvent* of A
- resolvent defined for $s \in \mathbb{C}$ *except* eigenvalues of A, $i.e.$, s such that $\textbf{det}(sI - A) = 0$
- $\mathcal{X}(s) = \textbf{det}(sI - A)$ is called the *characteristic polynomial* of A
- $\mathcal{X}(s)$ is a polynomial of degree n, with leading ($i.e.$, s^n) coefficient one
- roots of $\mathcal{X}$ are the eigenvalues of A

Matrix exponential

$(I - C)^{-1} = I + C + C^2 + C^3 + \cdots$ (if series converges)

- series expansion of resolvent:

  $$(sI - A)^{-1} = (1/s)(I - A/s)^{-1} = \frac{I}{s} + \frac{A}{s^2} + \frac{A^2}{s^3} + \cdots$$

  (valid for $|s|$ large enough) so

  $$\Phi(t) = \mathcal{L}^{-1}\left((sI - A)^{-1}\right) = I + tA + \frac{(tA)^2}{2!} + \cdots$$

- with this definition, state-transition matrix is

  $$\Phi(t) = \mathcal{L}^{-1}\left((sI - A)^{-1}\right) = e^{tA}$$

Discrete time systems

- the discrete-time LDS is $x(k+1) = Ax(k)$
- with solution $x(k) = A^k x(0)$
- The $\mathcal{Z}$-transform is $\hat{x} = \mathcal{Z}(x)$ where

  $$\hat{x}(z) = \sum_{k=0}^{\infty} x(k) z^{-k}$$

  for all $z \in \mathbb{C}$ with $|z|$ sufficiently large
- if $y(k) = x(k+1)$ then $\hat{y}(z) = z(\hat{x}(z) - x(0))$
- then $x(k+1) = Ax(k)$ implies $\hat{x}(z) = z(zI - A)^{-1}x(0)$ and

  $$\mathcal{Z}(A^k) = z(zI - A)^{-1}$$