

## Dynamic interpretation of eigenvectors

- ▶ invariant sets
- ▶ complex eigenvectors & invariant planes
- ▶ left eigenvectors
- ▶ modal form
- ▶ discrete-time stability

## Dynamic interpretation

suppose  $Av = \lambda v$ ,  $v \neq 0$

if  $\dot{x} = Ax$  and  $x(0) = v$ , then  $x(t) = e^{\lambda t}v$

several ways to see this, *e.g.*,

$$\begin{aligned}x(t) &= e^{tA}v = \left( I + tA + \frac{(tA)^2}{2!} + \dots \right) v \\ &= v + \lambda tv + \frac{(\lambda t)^2}{2!}v + \dots \\ &= e^{\lambda t}v\end{aligned}$$

(since  $(tA)^k v = (\lambda t)^k v$ )

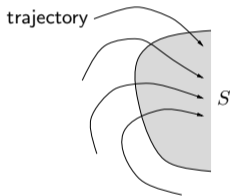
## Dynamic interpretation

- ▶ for  $\lambda \in \mathbb{C}$ , solution is complex (we'll interpret later); for now, assume  $\lambda \in \mathbb{R}$
- ▶ if initial state is an eigenvector  $v$ , resulting motion is very simple — always on the line spanned by  $v$
- ▶ solution  $x(t) = e^{\lambda t}v$  is called *mode* of system  $\dot{x} = Ax$  (associated with eigenvalue  $\lambda$ )
- ▶ for  $\lambda \in \mathbb{R}$ ,  $\lambda < 0$ , mode contracts or shrinks as  $t \uparrow$
- ▶ for  $\lambda \in \mathbb{R}$ ,  $\lambda > 0$ , mode expands or grows as  $t \uparrow$

## Invariant sets

a set  $S \subseteq \mathbb{R}^n$  is *invariant* under  $\dot{x} = Ax$  if whenever  $x(t) \in S$ , then  $x(\tau) \in S$  for all  $\tau \geq t$

*i.e.*: once trajectory enters  $S$ , it stays in  $S$



**vector field interpretation:** trajectories only cut *into*  $S$ , never out

## Invariant sets

suppose  $Av = \lambda v$ ,  $v \neq 0$ ,  $\lambda \in \mathbb{R}$

- ▶ line  $\{ tv \mid t \in \mathbb{R} \}$  is invariant  
(in fact, ray  $\{ tv \mid t > 0 \}$  is invariant)
- ▶ if  $\lambda < 0$ , line segment  $\{ tv \mid 0 \leq t \leq a \}$  is invariant

## Complex eigenvectors

suppose  $Av = \lambda v$ ,  $v \neq 0$ ,  $\lambda$  is complex

for  $a \in \mathbb{C}$ , (complex) trajectory  $ae^{\lambda t}v$  satisfies  $\dot{x} = Ax$

hence so does (real) trajectory

$$\begin{aligned}x(t) &= \Re (ae^{\lambda t}v) \\ &= e^{\sigma t} \begin{bmatrix} v_{\text{re}} & v_{\text{im}} \end{bmatrix} \begin{bmatrix} \cos \omega t & \sin \omega t \\ -\sin \omega t & \cos \omega t \end{bmatrix} \begin{bmatrix} \alpha \\ -\beta \end{bmatrix}\end{aligned}$$

where

$$v = v_{\text{re}} + iv_{\text{im}}, \quad \lambda = \sigma + i\omega, \quad a = \alpha + i\beta$$

because

$$(a + ib)(c + id) = \begin{bmatrix} 1 & -i \end{bmatrix} \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} c \\ -d \end{bmatrix}$$

## Complex eigenvectors

suppose  $Av = \lambda v$ ,  $v \neq 0$ ,  $\lambda$  is complex, then

$$x(t) = e^{\sigma t} \begin{bmatrix} v_{\text{re}} & v_{\text{im}} \end{bmatrix} \begin{bmatrix} \cos \omega t & \sin \omega t \\ -\sin \omega t & \cos \omega t \end{bmatrix} \begin{bmatrix} \alpha \\ -\beta \end{bmatrix}$$

- ▶ trajectory stays in *invariant plane*  $\text{span}\{v_{\text{re}}, v_{\text{im}}\}$
- ▶  $\sigma$  gives logarithmic growth/decay factor
- ▶  $\omega$  gives angular velocity of rotation in plane

## Dynamic interpretation: left eigenvectors

suppose  $w^T A = \lambda w^T$ ,  $w \neq 0$

then

$$\frac{d}{dt}(w^T x) = w^T \dot{x} = w^T A x = \lambda(w^T x)$$

*i.e.*,  $w^T x$  satisfies the DE  $d(w^T x)/dt = \lambda(w^T x)$

hence  $w^T x(t) = e^{\lambda t} w^T x(0)$

- ▶ even if trajectory  $x$  is complicated,  $w^T x$  is simple
- ▶ if, *e.g.*,  $\lambda \in \mathbb{R}$ ,  $\lambda < 0$ , halfspace  $\{ z \mid w^T z \leq a \}$  is invariant (for  $a \geq 0$ )
- ▶ for  $\lambda = \sigma + i\omega \in \mathbb{C}$ ,  $(\Re w)^T x$  and  $(\Im w)^T x$  both have form

$$e^{\sigma t} (\alpha \cos(\omega t) + \beta \sin(\omega t))$$

## Summary

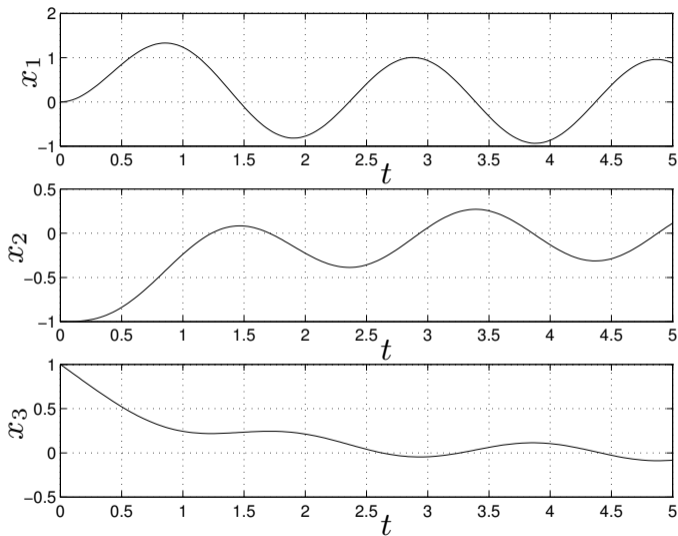
- ▶ *right eigenvectors* are initial conditions from which resulting motion is simple (*i.e.*, remains on line or in plane)
- ▶ *left eigenvectors* give linear functions of state that are simple, for any initial condition

## Example

$$\dot{x} = \begin{bmatrix} -1 & -10 & -10 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x$$

eigenvalues are  $-1, \pm i\sqrt{10}$

trajectory with  $x(0) = (0, -1, 1)$ :

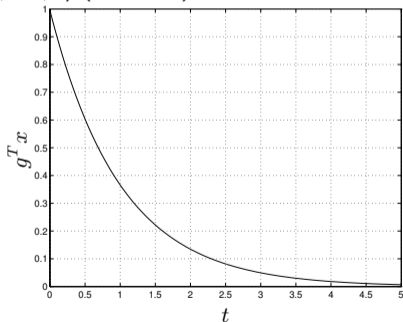


## Example

left eigenvector associated with eigenvalue  $-1$  is

$$g = \begin{bmatrix} 0.1 \\ 0 \\ 1 \end{bmatrix}$$

let's check  $g^T x(t)$  when  $x(0) = (0, -1, 1)$  (as above):



## Example

eigenvector associated with eigenvalue  $i\sqrt{10}$  is

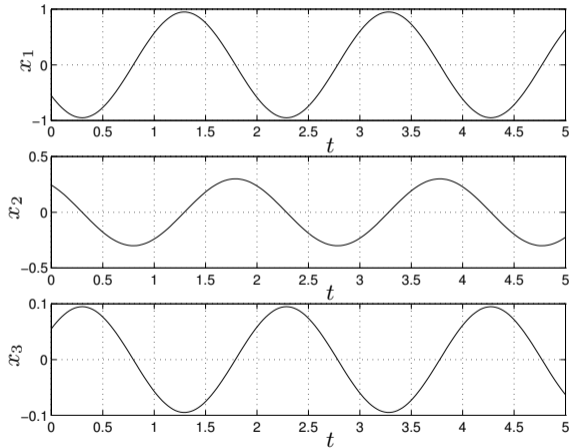
$$v = \begin{bmatrix} -0.554 + i0.771 \\ 0.244 + i0.175 \\ 0.055 - i0.077 \end{bmatrix}$$

so an invariant plane is spanned by

$$v_{\text{re}} = \begin{bmatrix} -0.554 \\ 0.244 \\ 0.055 \end{bmatrix}, \quad v_{\text{im}} = \begin{bmatrix} 0.771 \\ 0.175 \\ -0.077 \end{bmatrix}$$

## Example

for example, with  $x(0) = v_{re}$  we have



## Example: Markov chain

probability distribution satisfies  $p(t+1) = P^T p(t)$

$p_i(t) = \mathbf{Prob}(z(t) = i)$  so  $\sum_{i=1}^n p_i(t) = 1$

$P_{ij} = \mathbf{Prob}(z(t+1) = j \mid z(t) = i)$ , so  $\sum_{j=1}^n P_{ij} = 1$   
(such matrices are called *stochastic*)

rewrite as:

$$P\mathbf{1} = \mathbf{1}$$

where  $\mathbf{1} \in \mathbb{R}^n$  is the vector of all ones. *i.e.*,  $\mathbf{1}$  is a right eigenvector of  $P$  with e.v. 1

hence  $\det(I - P) = 0$ , so there is a left eigenvector  $w \neq 0$  with  $w^T P = w^T$

it can be shown that  $w$  can be chosen so that  $w_i \geq 0$ , hence we can normalize  $w$  so that  $\sum_{i=1}^n w_i = 1$

**interpretation:**  $w$  is an *equilibrium distribution*; *i.e.*, if  $p(0) = w$  then  $p(t) = w$  for all  $t \geq 0$

(if  $w$  is unique it is called the *steady-state distribution* of the Markov chain)

## Modal form

- ▶ suppose  $A$  is diagonalizable by  $T$
- ▶ define new coordinates by  $x = T\tilde{x}$ , so

$$T\dot{\tilde{x}} = AT\tilde{x} \quad \Leftrightarrow \quad \dot{\tilde{x}} = T^{-1}AT\tilde{x} \quad \Leftrightarrow \quad \dot{\tilde{x}} = \Lambda\tilde{x}$$

- ▶ in new coordinate system, system is diagonal (decoupled)
- ▶ trajectories consist of  $n$  independent modes, *i.e.*,

$$\tilde{x}_i(t) = e^{\lambda_i t} \tilde{x}_i(0)$$

hence the name *modal form*

## Real modal form

if  $A$  is diagonalizable, and eigenvalues (hence  $T$ ) are complex, system can be put in *real modal form*:

$$S^{-1}AS = \mathbf{diag}(\Lambda_r, M_{r+1}, M_{r+3}, \dots, M_{n-1})$$

where  $\Lambda_r = \mathbf{diag}(\lambda_1, \dots, \lambda_r)$  are the real eigenvalues, and

$$M_j = \begin{bmatrix} \sigma_j & \omega_j \\ -\omega_j & \sigma_j \end{bmatrix}, \quad \lambda_j = \sigma_j + i\omega_j, \quad j = r+1, r+3, \dots, n$$

where  $\lambda_j$  are the complex eigenvalues (one from each conjugate pair)

## Real modal form

- ▶ Suppose  $AQ = QD$  where

$$Q = [v \quad \bar{v}] \quad D = \begin{bmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{bmatrix} \quad \lambda = \sigma + i\omega$$

- ▶ use change of variables  $Z = \begin{bmatrix} \frac{1}{2} & -\frac{i}{2} \\ \frac{1}{2} & \frac{i}{2} \end{bmatrix}$

- ▶ notice that

$$QZ = [\Re v \quad \Im v] \quad Z^{-1}DZ = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}$$

- ▶ then  $AQZ = QZZ^{-1}DZ$ , which is

$$A [\Re v \quad \Im v] = [\Re v \quad \Im v] \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}$$

## Diagonalization and exponential

$$\begin{aligned}e^A &= I + A + A^2/2! + \dots \\&= I + T\Lambda T^{-1} + (T\Lambda T^{-1})^2/2! + \dots \\&= T(I + \Lambda + \Lambda^2/2! + \dots)T^{-1} \\&= Te^\Lambda T^{-1} \\&= T \mathbf{diag}(e^{\lambda_1}, \dots, e^{\lambda_n})T^{-1}\end{aligned}$$

## Solution via diagonalization

assume  $A$  is diagonalizable

consider LDS  $\dot{x} = Ax$ , with  $T^{-1}AT = \Lambda$

then

$$\begin{aligned}x(t) &= e^{tA}x(0) \\ &= Te^{\Lambda t}T^{-1}x(0) \\ &= \sum_{i=1}^n e^{\lambda_i t} (w_i^T x(0)) v_i\end{aligned}$$

thus: any trajectory can be expressed as linear combination of modes

## Interpretation

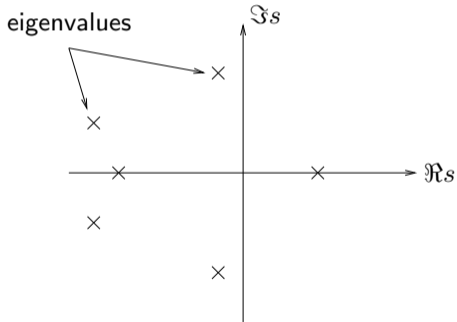
- ▶ (left eigenvectors) decompose initial state  $x(0)$  into modal components  $w_i^T x(0)$
- ▶  $e^{\lambda_i t}$  term propagates  $i$ th mode forward  $t$  seconds
- ▶ reconstruct state as linear combination of (right) eigenvectors

## Qualitative behavior of $x(t)$

- ▶ eigenvalues give exponents that can occur in exponentials
- ▶ real eigenvalue  $\lambda$  corresponds to an exponentially decaying or growing term  $e^{\lambda t}$  in solution
- ▶ complex eigenvalue  $\lambda = \sigma + i\omega$  corresponds to decaying or growing sinusoidal term  $e^{\sigma t} \cos(\omega t + \phi)$  in solution

## Qualitative behavior of $x(t)$

- ▶  $\Re\lambda_j$  gives exponential growth rate (if  $> 0$ ), or exponential decay rate (if  $< 0$ ) of term
- ▶  $\Im\lambda_j$  gives frequency of oscillatory term (if  $\neq 0$ )



## Application

- ▶ for what  $x(0)$  do we have  $x(t) \rightarrow 0$  as  $t \rightarrow \infty$ ?
- ▶ suppose  $A$  is diagonalizable, and divide eigenvalues into those with negative real parts

$$\Re \lambda_1 < 0, \dots, \Re \lambda_s < 0,$$

and the others,

$$\Re \lambda_{s+1} \geq 0, \dots, \Re \lambda_n \geq 0$$

- ▶ since

$$x(t) = \sum_{i=1}^n e^{\lambda_i t} (w_i^T x(0)) v_i$$

the condition for  $x(t) \rightarrow 0$  is

$$x(0) \in \mathbf{span}\{v_1, \dots, v_s\},$$

or equivalently,

$$w_i^T x(0) = 0, \quad i = s + 1, \dots, n$$