

Jordan canonical form

- ▶ Jordan canonical form
- ▶ generalized modes
- ▶ Cayley-Hamilton theorem

Jordan canonical form

any matrix $A \in \mathbb{R}^{n \times n}$ can be put in *Jordan canonical form* by a similarity transformation, i.e.

$$T^{-1}AT = J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_q \end{bmatrix}$$

where

$$J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix} \in \mathbb{C}^{n_i \times n_i}$$

is called a *Jordan block* of size n_i with eigenvalue λ_i (so $n = \sum_{i=1}^q n_i$)

- ▶ J is upper bidiagonal
- ▶ J diagonal is the special case of n Jordan blocks of size $n_i = 1$
- ▶ Jordan form is unique (up to permutations of the blocks)
- ▶ can have multiple blocks with same eigenvalue

Jordan canonical form

note: JCF is a *conceptual tool*, never used in numerical computations!

$$\mathcal{X}(s) = \mathbf{det}(sI - A) = (s - \lambda_1)^{n_1} \cdots (s - \lambda_q)^{n_q}$$

hence distinct eigenvalues $\Rightarrow n_i = 1 \Rightarrow A$ diagonalizable

dim null $(\lambda I - A)$ is the number of Jordan blocks with eigenvalue λ

more generally,

$$\mathbf{dim null}(\lambda I - A)^k = \sum_{\lambda_i = \lambda} \mathbf{min}\{k, n_i\}$$

so from **dim null** $(\lambda I - A)^k$ for $k = 1, 2, \dots$ we can determine the sizes of the Jordan blocks associated with λ

Jordan canonical form

- ▶ factor out T and T^{-1} , $\lambda I - A = T(\lambda I - J)T^{-1}$
- ▶ for the i th block: write $J_i = \lambda_i I + N$ where N is nilpotent (ones on first upper diagonal)
- ▶ for the same block ($\lambda = \lambda_i$): $\lambda_i I - J_i = -N$, so

$$-N = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad N^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad N^3 = 0$$

hence $(\lambda_i I - J_i)^k = (-N)^k = 0$ for $k \geq n_i$

Jordan canonical form

- for a different block ($\lambda \neq \lambda_j$): let $\mu = \lambda - \lambda_j$, then $\lambda I - J_j = \mu I - N$
by binomial theorem (since μI and N commute):

$$(\mu I - N)^k = \sum_{m=0}^{\min(k, n_j-1)} \binom{k}{m} (-1)^m \mu^{k-m} N^m$$

for a block of size 3:

$$(\mu I - N)^k = \begin{bmatrix} \mu^k & -k\mu^{k-1} & \binom{k}{2}\mu^{k-2} \\ 0 & \mu^k & -k\mu^{k-1} \\ 0 & 0 & \mu^k \end{bmatrix}$$

Generalized eigenvectors

suppose $T^{-1}AT = J = \mathbf{diag}(J_1, \dots, J_q)$

express T as

$$T = [T_1 \ T_2 \ \cdots \ T_q]$$

where $T_i \in \mathbb{C}^{n \times n_i}$ are the columns of T associated with i th Jordan block J_i

we have $AT_i = T_i J_i$

let $T_i = [v_{i1} \ v_{i2} \ \cdots \ v_{in_i}]$

then we have:

$$Av_{i1} = \lambda_i v_{i1},$$

i.e., the first column of each T_i is an eigenvector associated with e.v. λ_i

for $j = 2, \dots, n_i,$

$$Av_{ij} = v_{i \ j-1} + \lambda_i v_{ij}$$

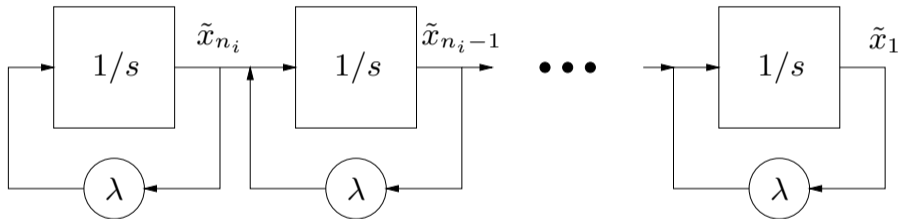
the vectors v_{i1}, \dots, v_{in_i} are sometimes called *generalized eigenvectors*

Jordan form LDS

consider LDS $\dot{x} = Ax$

by change of coordinates $x = T\tilde{x}$, can put into form $\dot{\tilde{x}} = J\tilde{x}$

system is decomposed into independent 'Jordan block systems' $\dot{\tilde{x}}_i = J_i\tilde{x}_i$



Jordan blocks are sometimes called Jordan chains

(block diagram shows why)

Jordan chains: why $t^p e^{\lambda t}$

a Jordan block of size 3 gives the chain of ODEs:

$$\dot{\tilde{x}}_3 = \lambda \tilde{x}_3, \quad \dot{\tilde{x}}_2 = \lambda \tilde{x}_2 + \tilde{x}_3, \quad \dot{\tilde{x}}_1 = \lambda \tilde{x}_1 + \tilde{x}_2$$

solve from the bottom up, with $\tilde{x}_3(0) = 1$, $\tilde{x}_2(0) = \tilde{x}_1(0) = 0$:

$$\tilde{x}_3(t) = e^{\lambda t}$$

$$\tilde{x}_2(t) = e^{\lambda t} \int_0^t \underbrace{e^{-\lambda \tau} e^{\lambda \tau}}_{=1} d\tau = t e^{\lambda t}$$

$$\tilde{x}_1(t) = e^{\lambda t} \int_0^t \underbrace{e^{-\lambda \tau} \tau e^{\lambda \tau}}_{=\tau} d\tau = \frac{t^2}{2} e^{\lambda t}$$

at each stage, the integrating factor $e^{-\lambda \tau}$ cancels the forcing $e^{\lambda \tau}$ (first-order 'resonance'), leaving a polynomial to integrate — which raises the degree by one

a chain of length k produces terms up to $t^{k-1} e^{\lambda t}$

Resolvent, exponential of Jordan block

- ▶ resolvent of $k \times k$ Jordan block with eigenvalue λ :

$$\begin{aligned}(sI - J_\lambda)^{-1} &= \begin{bmatrix} s - \lambda & -1 & & \\ & s - \lambda & \ddots & \\ & & \ddots & -1 \\ & & & s - \lambda \end{bmatrix}^{-1} \\ &= \begin{bmatrix} (s - \lambda)^{-1} & (s - \lambda)^{-2} & \cdots & (s - \lambda)^{-k} \\ & (s - \lambda)^{-1} & \cdots & (s - \lambda)^{-k+1} \\ & & \ddots & \vdots \\ & & & (s - \lambda)^{-1} \end{bmatrix} \\ &= (s - \lambda)^{-1}I + (s - \lambda)^{-2}F_1 + \cdots + (s - \lambda)^{-k}F_{k-1}\end{aligned}$$

where F_i is the matrix with ones on the i th upper diagonal

- ▶ recall $e^{\lambda t} t^{n-1} / (n-1)!$ has Laplace transform $(s - \lambda)^{-n}$

Exponential of Jordan block

- ▶ consider a single jordan block $J = \lambda I + N$ of size $k \times k$
- ▶ N is the nilpotent matrix with ones on the 1st upper diagonal
- ▶ the exponential is

$$\begin{aligned}e^{tJ} &= e^{(\lambda I + N)t} \\ &= e^{t\lambda} e^{tN} \\ &= e^{t\lambda} \left(I + tN + \dots + (t^{k-1}/(k-1)!)N^{k-1} \right) \\ &= e^{t\lambda} \begin{bmatrix} 1 & t & \dots & t^{k-1}/(k-1)! \\ & 1 & \dots & t^{k-2}/(k-2)! \\ & & \ddots & \vdots \\ & & & 1 \end{bmatrix}\end{aligned}$$

Jordan blocks yield terms of form $t^p e^{t\lambda}$ in e^{tA}

Stability

- ▶ we say system $\dot{x} = Ax$ is *stable* if $e^{tA} \rightarrow 0$ as $t \rightarrow \infty$
 - ▶ state $x(t)$ converges to 0, as $t \rightarrow \infty$, no matter what $x(0)$ is
 - ▶ all trajectories of $\dot{x} = Ax$ converge to 0 as $t \rightarrow \infty$

- ▶ $\dot{x} = Ax$ is stable if and only if all eigenvalues of A have negative real part:

$$\Re \lambda_i < 0, \quad i = 1, \dots, n$$

from the expression for e^{tJ}

Generalized modes

consider $\dot{x} = Ax$, with

$$x(0) = a_1 v_{i1} + \cdots + a_{n_i} v_{in_i} = T_i a$$

then $x(t) = T e^{Jt} \tilde{x}(0) = T_i e^{J_i t} a$

- ▶ trajectory stays in span of generalized eigenvectors
- ▶ coefficients have form $p(t)e^{\lambda t}$, where p is polynomial
- ▶ such solutions are called *generalized modes* of the system

with general $x(0)$ we can write

$$x(t) = e^{tA} x(0) = T e^{tJ} T^{-1} x(0) = \sum_{i=1}^q T_i e^{tJ_i} (S_i^T x(0))$$

where

$$T^{-1} = \begin{bmatrix} S_1^T \\ \vdots \\ S_q^T \end{bmatrix}$$

hence: all solutions of $\dot{x} = Ax$ are linear combinations of (generalized) modes

Powers of a Jordan block

- ▶ consider a single jordan block $J = \lambda I + N$

$$\begin{aligned}(\lambda I + N)^k &= \sum_{j=0}^k \binom{k}{j} \lambda^k N^{k-j} \\ &= \begin{bmatrix} \lambda^k & k\lambda^{k-1} & \frac{k(k-1)}{2}\lambda^{k-2} & \dots \\ & \lambda^k & \dots & \\ & & \ddots & \end{bmatrix}\end{aligned}$$

- ▶ $x(t+1) = Ax(t)$ is stable (*i.e.*, $A^k \rightarrow 0$ as $k \rightarrow \infty$) if and only if all eigenvalues of A have magnitude less than one

Cayley-Hamilton theorem

if $p(s) = a_0 + a_1s + \cdots + a_k s^k$ is a polynomial and $A \in \mathbb{R}^{n \times n}$, we define

$$p(A) = a_0I + a_1A + \cdots + a_k A^k$$

Cayley-Hamilton theorem: for any $A \in \mathbb{R}^{n \times n}$ we have $\mathcal{X}(A) = 0$, where $\mathcal{X}(s) = \det(sI - A)$

example: with $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ we have $\mathcal{X}(s) = s^2 - 5s - 2$, so

$$\begin{aligned}\mathcal{X}(A) &= A^2 - 5A - 2I \\ &= \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} - 5 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - 2I \\ &= 0\end{aligned}$$

Cayley-Hamilton theorem

corollary: for every $p \in \mathbb{Z}_+$, we have

$$A^p \in \text{span} \{ I, A, A^2, \dots, A^{n-1} \}$$

(and if A is invertible, also for $p \in \mathbb{Z}$)

i.e., every power of A can be expressed as linear combination of I, A, \dots, A^{n-1}

proof: divide $\mathcal{X}(s)$ into s^p to get $s^p = q(s)\mathcal{X}(s) + r(s)$

$r = \alpha_0 + \alpha_1 s + \dots + \alpha_{n-1} s^{n-1}$ is remainder polynomial

then

$$A^p = q(A)\mathcal{X}(A) + r(A) = r(A) = \alpha_0 I + \alpha_1 A + \dots + \alpha_{n-1} A^{n-1}$$

Cayley-Hamilton theorem

for $p = -1$: rewrite C-H theorem

$$\mathcal{X}(A) = A^n + a_{n-1}A^{n-1} + \cdots + a_0I = 0$$

as

$$I = A \left(-(a_1/a_0)I - (a_2/a_0)A - \cdots - (1/a_0)A^{n-1} \right)$$

(A is invertible $\Leftrightarrow a_0 \neq 0$) so

$$A^{-1} = -(a_1/a_0)I - (a_2/a_0)A - \cdots - (1/a_0)A^{n-1}$$

i.e., inverse is linear combination of A^k , $k = 0, \dots, n-1$

for $p = -2, -3, \dots$, use induction:

$$A^{p-1} = -(a_1/a_0)A^p - (a_2/a_0)A^{p+1} - \cdots - (1/a_0)A^{p+n}$$

if A^p, \dots, A^{p+n} are linear combinations of A^k , $k = 0, \dots, n-1$, so is A^{p-1}

Proof of C-H theorem

first assume A is diagonalizable: $T^{-1}AT = \Lambda$

$$\mathcal{X}(s) = (s - \lambda_1) \cdots (s - \lambda_n)$$

since

$$\mathcal{X}(A) = \mathcal{X}(T\Lambda T^{-1}) = T\mathcal{X}(\Lambda)T^{-1}$$

it suffices to show $\mathcal{X}(\Lambda) = 0$

$$\begin{aligned}\mathcal{X}(\Lambda) &= (\Lambda - \lambda_1 I) \cdots (\Lambda - \lambda_n I) \\ &= \mathbf{diag}(0, \lambda_2 - \lambda_1, \dots, \lambda_n - \lambda_1) \cdots \mathbf{diag}(\lambda_1 - \lambda_n, \dots, \lambda_{n-1} - \lambda_n, 0) \\ &= 0\end{aligned}$$

Proof of C-H theorem

now let's do general case: $T^{-1}AT = J$

$$\mathcal{X}(s) = (s - \lambda_1)^{n_1} \cdots (s - \lambda_q)^{n_q}$$

suffices to show $\mathcal{X}(J_i) = 0$

$$\mathcal{X}(J_i) = (J_i - \lambda_1 I)^{n_1} \cdots \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ & & & \ddots \\ & & & & \ddots \end{bmatrix}}_{(J_i - \lambda_i I)^{n_i}} \cdots (J_i - \lambda_q I)^{n_q} = 0$$