Continuous-time linear systems

- the state and causality
- variation of constants
- impulse and step responses
- transfer functions
- examples

CT System Definition, State, and Solution

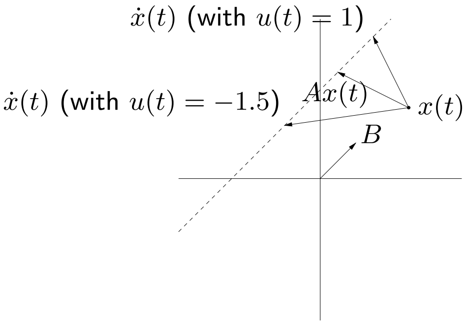
Inputs & outputs

recall continuous-time time-invariant LDS has form

$$\dot{x} = Ax + Bu, \qquad y = Cx + Du$$

- Ax is called the *drift term* (of $\dot{x}$)
- Bu is called the input term (of $\dot{x}$)

picture, with $B \in \mathbb{R}^{2\times 1}$:



Interpretations

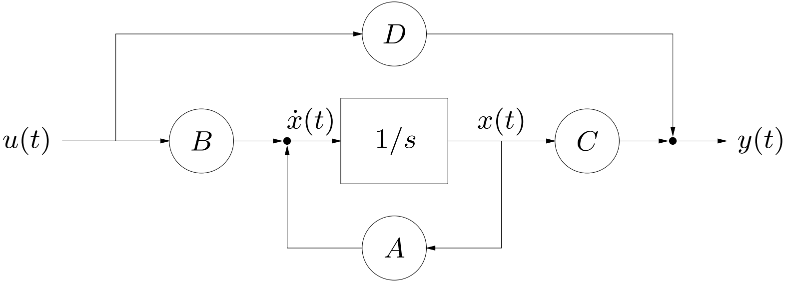
write $\dot{x} = Ax + b_1 u_1 + \cdots + b_m u_m$, where $B = \begin{bmatrix} b_1 & \cdots & b_m \end{bmatrix}$

- state derivative is sum of autonomous term (Ax) and one term per input $(b_i u_i)$
- each input u_i gives another degree of freedom for $\dot{x}$ (assuming columns of B independent)

write $\dot{x} = Ax + Bu$ as $\dot{x}_i = \tilde{a}_i^\mathsf{T} x + \tilde{b}_i^\mathsf{T} u$, where $\tilde{a}_i^\mathsf{T}$, $\tilde{b}_i^\mathsf{T}$ are the rows of A, B

- ith state derivative is linear function of state x and input u

Block diagram



- A_{ij} is gain factor from state x_j into integrator i
- B_{ij} is gain factor from input u_j into integrator i
- C_{ij} is gain factor from state x_j into output y_i
- D_{ij} is gain factor from input u_j into output y_i

Solution

- we would like to solve $\dot{x} = Ax + Bu$
- use the change of variables $x(t) = e^{tA}y(t)$ (called an *integrating factor*)
- then substituting gives $e^{tA}\dot{y} + Ae^{tA}y = Ae^{tA}y + Bu$ and so $e^{tA}\dot{y} = u$
- hence $\dot{y} = e^{-tA}u$ which we integrate

$$y(t) - y(0) = \int_0^t e^{-A\tau}u(\tau)\,d\tau$$

and so

$$x(t) = e^{tA}x(0) + \int_0^t e^{(t-\tau)A}Bu(\tau)\,d\tau$$

Response to input

- the solution to $\dot{x} = Ax + Bu$ is

$$x(t) = e^{tA}x(0) + \int_0^t e^{(t-\tau)A}Bu(\tau)\,d\tau$$

- $e^{tA}x(0)$ is the unforced or autonomous response
- $e^{tA}B$ is called the input-to-state impulse response or impulse matrix

Idea of state

$x(t)$ is called *state* of system at time t since:

- future output depends only on current state and future input
- future output depends on past input only through current state
- state summarizes effect of past inputs on future output
- state is bridge between past inputs and future outputs

Causality

interpretation of

$$x(t) = e^{tA}x(0) + \int_0^t e^{(t-\tau)A}Bu(\tau)\,d\tau$$

$$y(t) = Ce^{tA}x(0) + \int_0^t Ce^{(t-\tau)A}Bu(\tau)\,d\tau + Du(t)$$

for $t \geq 0$:

current state ($x(t)$) and output ($y(t)$) depend on *past* input ($u(\tau)$ for $\tau \leq t$)

$i.e.$, mapping from input to state and output is *causal* (with fixed *initial* state)

now consider fixed *final* state $x(T)$: for $t \leq T$,

$$x(t) = e^{(t-T)A}x(T) + \int_T^t e^{(t-\tau)A}Bu(\tau)\,d\tau,$$

$i.e.$, current state (and output) depend on future input!

so for fixed final condition, same system is anti-causal

CT Impulse/Step Response and Transfer Function

Impulse response

impulse response $h(t) = Ce^{tA}B + D\delta(t)$

with $x(0) = 0$, $y = h * u$, $i.e.$,

$$y_i(t) = \sum_{j=1}^{m} \int_0^t h_{ij}(t-\tau)u_j(\tau)\, d\tau$$

interpretations:

- $h_{ij}(t)$ is impulse response from jth input to ith output
- $h_{ij}(t)$ gives $y_i(t)$ when $u(t) = e_j\delta(t)$
- $h_{ij}(\tau)$ shows how dependent output i is, on what input j was, τ seconds ago
- i indexes output; j indexes input; τ indexes time lag

Step response

the *step response* or *step matrix* is given by

$$s(t) = \int_0^t h(\tau)\, d\tau$$

interpretations:

- $s_{ij}(t)$ is step response from jth input to ith output
- $s_{ij}(t)$ gives y_i when $u = e_j$ for $t \geq 0$

for invertible A, we have

$$s(t) = CA^{-1}\left(e^{tA} - I\right)B + D$$

DC or static gain matrix

- DC gain describes system under *static* conditions, $i.e.$, x, u, y constant:

  $$0 = \dot{x} = Ax + Bu, \qquad y = Cx + Du$$

  eliminate x to get $y = H_0 u$ where

  $$H_0 = -CA^{-1}B + D = H(0)$$

  ($i.e.$, the transfer function evaluated at $s = 0$)

- if system is stable,

  $$H_0 = \int_0^\infty h(t)\, dt = \lim_{t \to \infty} s(t)$$

  (recall: $H(s) = \int_0^\infty e^{-st} h(t)\, dt$, $s(t) = \int_0^t h(\tau)\, d\tau$)

  if $u(t) \to u_\infty \in \mathbb{R}^m$, then $y(t) \to y_\infty \in \mathbb{R}^p$ where $y_\infty = H_0 u_\infty$

Transfer function

- take Laplace transform of $\dot{x} = Ax + Bu$:

$$s\hat{x}(s) - x(0) = A\hat{x}(s) + B\hat{u}(s)$$

- hence

$$\hat{x}(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}B\hat{u}(s)$$

- $e^{tA}x(0)$ is the unforced or autonomous response
- $(sI - A)^{-1}B$ is called the *input-to-state transfer function* or *transfer matrix*

Transfer function

- with $y = Cx + Du$ we have

$$\hat{y}(s) = C(sI - A)^{-1}x(0) + (C(sI - A)^{-1}B + D)\hat{u}(s)$$

- $H(s) = C(sI - A)^{-1}B + D$ is called the *transfer function* or *transfer matrix*
- with zero initial condition we have:

$$\hat{y}(s) = \hat{H}(s)\hat{u}(s)$$

- H_{ij} is transfer function from input u_j to output y_i

Partial fraction expansion

for a SISO system with diagonalizable A and distinct eigenvalues $\lambda_1, \ldots, \lambda_n$:

$$G(s) = C(sI - A)^{-1}B = \sum_{i=1}^{n} \frac{r_i}{s - \lambda_i}$$

where $r_i = (Cv_i)(w_i^T B)$ are the *residues*

(v_i, w_i are right and left eigenvectors of A)

inverse Laplace transform gives the impulse response:

$$h(t) = Ce^{At}B = \sum_{i=1}^{n} r_i e^{\lambda_i t}, \quad t > 0$$

- each mode $e^{\lambda_i t}$ contributes to $h(t)$ with weight r_i
- r_i depends on how strongly mode i couples to the input ($w_i^T b$) and to the output ($c^T v_i$)
- if $r_i = 0$, eigenvalue λ_i does not appear in $G(s)$ (pole-zero cancellation)

Partial fraction expansion: MIMO

for MIMO with diagonalizable A:

$$G(s) = C(sI - A)^{-1}B + D = \sum_{i=1}^{n} \frac{R_i}{s - \lambda_i} + D$$

where $R_i = (Cv_i)(w_i^\top B) \in \mathbb{R}^{p\times m}$ are the *residue matrices*

- R_i is rank one (outer product of column and row vectors)
- $R_i = 0$ means eigenvalue λ_i is invisible in $G(s)$
- for repeated eigenvalues (Jordan blocks), additional terms $R_k/(s-\lambda)^k$ appear (cf. resolvent of Jordan block from lecture 4)

example: $A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 \end{bmatrix}$, $D = 0$

$$G(s) = \frac{1}{s+1} + \frac{1}{s+2} = \frac{2s+3}{(s+1)(s+2)}$$

Frequency response

if system is stable, the *frequency response* is $G(j\omega)$ for $\omega \in \mathbb{R}$

physical meaning: if $u(t) = a\sin(\omega t)$, then the steady-state output is

$$y_{\rm ss}(t) = a\,|G(j\omega)|\sin(\omega t + \angle G(j\omega))$$

- $|G(j\omega)|$ is the *gain* at frequency ω
- $\angle G(j\omega)$ is the *phase shift* at frequency ω
- MIMO case: G_{ij} gives gain and phase shift for output i when sinusoid applied at input j

the H_∞ *norm* is the worst-case gain over all frequencies and directions:

$$||G||_\infty = \sup_\omega \bar{\sigma}(G(j\omega))$$

Frequency response example

$$G(s) = \frac{1}{s+1}: \qquad G(j\omega) = \frac{1}{j\omega + 1}$$

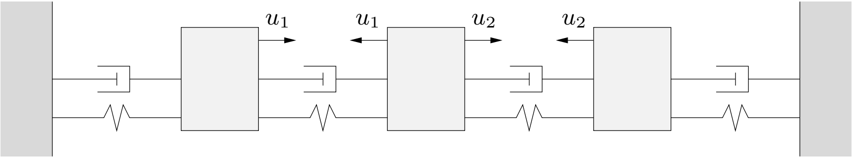
$$|G(j\omega)| = \frac{1}{\sqrt{1+\omega^2}}, \qquad \angle G(j\omega) = -\arctan(\omega)$$

- at $\omega = 0$: gain $= 1$, phase $= 0$ (consistent with DC gain $G(0) = 1$)
- at $\omega = 1$: gain $= 1/\sqrt{2}$, phase $= -45^\circ$
- as $\omega \to \infty$: gain $\to 0$, phase $\to -90^\circ$
- $\|G\|_\infty = 1$ (achieved at $\omega = 0$)

sinusoidal steady state: $u(t) = \sin(t)$ gives $y_{\mathrm{ss}}(t) = \frac{1}{\sqrt{2}} \sin(t - 45^\circ)$

CT Examples

Mass-spring example



- unit masses, springs, dampers
- u_1 is tension between 1st & 2nd masses
- u_2 is tension between 2nd & 3rd masses
- $y \in \mathbb{R}^3$ is displacement of masses 1,2,3
- $x = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$

Mass-spring example

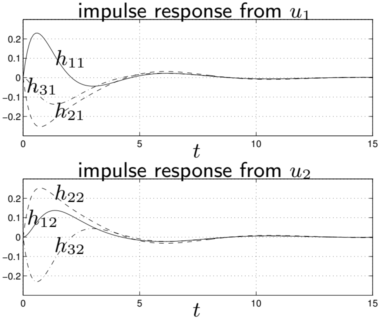
system is:

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & -2 & 1 & 0 \\ 1 & -2 & 1 & 1 & -2 & 1 \\ 0 & 1 & -2 & 0 & 1 & -2 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

eigenvalues of A are

$$-1.71 \pm i0.71, \qquad -1.00 \pm i1.00, \qquad -0.29 \pm i0.71$$

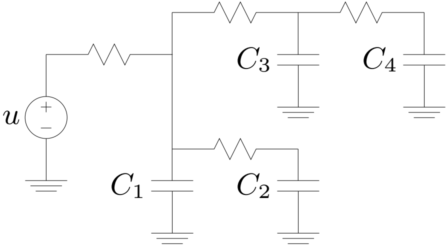
Example: Impulse response



roughly speaking:

- impulse at u_1 affects third mass less than other two
- impulse at u_2 affects first mass later than other two

Circuit example



- $u(t) \in \mathbb{R}$ is input (drive) voltage
- x_i is voltage across C_i
- output is state: $y = x$
- unit resistors, unit capacitors
- step response matrix shows delay to each node

Circuit example

system is

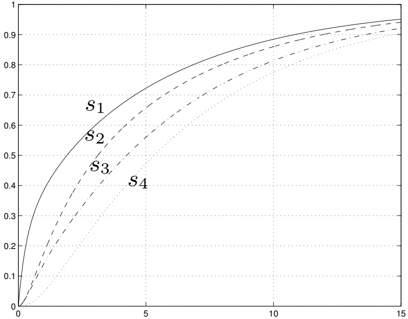
$$\dot{x} = \begin{bmatrix} -3 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u, \qquad y = x$$

eigenvalues of A are

$$-0.17, \qquad -0.66, \qquad -2.21, \qquad -3.96$$

Circuit example

step response matrix $s(t) \in \mathbb{R}^{4\times 1}$:



- shortest delay to x_1; longest delay to x_4
- delays consistent with slowest ($i.e.$, dominant) eigenvalue -0.17

DC gain matrix

DC gain matrix for spring-mass example:

$$H_0 = \begin{bmatrix} 1/4 & 1/4 \\ -1/2 & 1/2 \\ -1/4 & -1/4 \end{bmatrix}$$

DC gain matrix for circuit example:

$$H_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

(do these make sense?)