Discrete-time systems and standard forms

- discretization of continuous-time systems
- discrete-time systems and solutions
- discrete-time transfer functions
- change of coordinates and standard forms
- examples

Discretization

Discretization with piecewise constant inputs

linear system $\dot{x} = Ax + Bu$, $y = Cx + Du$

suppose $u_d : \mathbb{Z}_+ \to \mathbb{R}^m$ is a sequence, and

$$u(t) = u_d(k) \qquad \text{for} \quad kh \leq t < (k+1)h, \quad k = 0, 1, \ldots$$

define sequences

$$x_d(k) = x(kh), \qquad y_d(k) = y(kh), \qquad k = 0, 1, \ldots$$

- $h > 0$ is called the *sample interval* (for x and y) or *update interval* (for u)
- u is piecewise constant (called *zero-order-hold*)
- x_d, y_d are sampled versions of x, y

Discretization with piecewise constant inputs

$$
\begin{aligned}
x_d(k+1) &= x((k+1)h) \\
&= e^{hA}x(kh) + \int_0^h e^{\tau A} B u((k+1)h - \tau)\, d\tau \\
&= e^{hA}x_d(k) + \left(\int_0^h e^{\tau A}\, d\tau\right) B\ u_d(k)
\end{aligned}
$$

x_d, u_d, and y_d satisfy discrete-time LDS equations

$$x_d(k+1) = A_d x_d(k) + B_d u_d(k), \qquad y_d(k) = C_d x_d(k) + D_d u_d(k)$$

where

$$A_d = e^{hA}, \qquad B_d = \left(\int_0^h e^{\tau A}\, d\tau\right) B, \qquad C_d = C, \qquad D_d = D$$

called *discretized system*. If A is invertible, we can express integral as

$$\int_0^h e^{\tau A}\, d\tau = A^{-1}\left(e^{hA} - I\right)$$

Stability of discretization

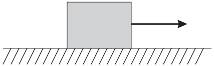
stability: if eigenvalues of A are $\lambda_1, \ldots, \lambda_n$, then eigenvalues of A_d are $e^{h\lambda_1}, \ldots, e^{h\lambda_n}$

discretization preserves stability properties since

$$\Re\lambda_i < 0 \quad \Leftrightarrow \quad \left|e^{h\lambda_i}\right| < 1$$

for $h > 0$

Example: Force on mass



Newton's law gives continuous-time LDS

$$
\begin{aligned}
\dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\
y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)
\end{aligned}
$$

let's compute the discretization

$$
\begin{aligned}
A_d &= e^{Ah} \\
&= I + Ah + \frac{1}{2}A^2h^2 + \cdots \\
&= I + Ah \\
&= \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}
\end{aligned}
$$

Example: Force on mass

$$
\begin{aligned}
B_d &= \int_0^h e^{As} B \, ds \\
&= \int_0^h \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} ds \\
&= \int_0^h \begin{bmatrix} s \\ 1 \end{bmatrix} ds = \begin{bmatrix} \frac{1}{2}h^2 \\ h \end{bmatrix}
\end{aligned}
$$

so the discretization is

$$
\begin{aligned}
x_d(k+1) &= \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} x_d(k) + \begin{bmatrix} \frac{1}{2}h^2 \\ h \end{bmatrix} u_d(k) \\
y_d(k) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x_d(k)
\end{aligned}
$$

Extensions and variations

- *offsets:* updates for u and sampling of x, y are offset in time
- *multirate:* u_i updated, y_i sampled at different intervals

  (usually integer multiples of a common interval h)

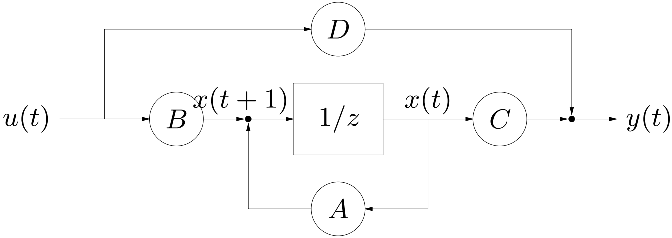
both very common in practice

Discrete-Time Systems

Discrete-time systems

discrete-time LDS:

$$x(t+1) = Ax(t) + Bu(t), \qquad y(t) = Cx(t) + Du(t)$$



- only difference w/cts-time: z instead of s
- interpretation of z^{-1} block:
  - unit delay (shifts sequence back in time one epoch)
  - latch (plus small delay to avoid race condition)

Discrete-time systems

discrete-time LDS:

$$x(t+1) = Ax(t) + Bu(t), \qquad y(t) = Cx(t) + Du(t)$$

we have:

$$x(1) = Ax(0) + Bu(0),$$

$$\begin{aligned} x(2) &= Ax(1) + Bu(1) \\ &= A^2x(0) + ABu(0) + Bu(1), \end{aligned}$$

and in general, for $t \in \mathbb{Z}_+$,

$$x(t) = A^t x(0) + \sum_{\tau=0}^{t-1} A^{(t-1-\tau)} Bu(\tau)$$

Discrete-time systems

Solution is

$$x(t) = A^t x(0) + \sum_{\tau=0}^{t-1} A^{(t-1-\tau)} B u(\tau)$$

write this as

$$y(t) = CA^t x(0) + H * u$$

where $*$ is discrete-time convolution

$$y(t) = CA^t x(0) + \sum_{\tau=0}^{t} H(t-\tau) u(\tau)$$

and

$$H(t) = \begin{cases} D, & t = 0 \\ CA^{t-1}B, & t > 0 \end{cases}$$

is the impulse response

Block Toeplitz matrices

we have

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ \vdots \\ y(t) \end{bmatrix} = \begin{bmatrix} D & & & & \\ CB & D & & & \\ CAB & CB & D & & \\ \vdots & & & \ddots & \\ CA^{t-1}B & CA^{t-2}B & \dots & CB & D \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ u(2) \\ \vdots \\ u(t) \end{bmatrix} + \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^t \end{bmatrix} x(0)$$

- this matrix gives the output sequence $y(0), y(1), \dots$ in terms of the input sequence $u(0), u(1), \dots$ and the initial state $x(0)$
- *block Toeplitz* means blocks are constant along diagonals from top-left to bottom right
- we can use this to find controllers and estimators

Discrete-time transfer function

- recall: if $v(t) = w(t+1)$ then $\hat{v}(z) = z\hat{w}(z) - zw(0)$
- take $\mathcal{Z}$-transform of system equations

$$x(t+1) = Ax(t) + Bu(t), \qquad y(t) = Cx(t) + Du(t)$$

  yields

$$z\hat{x}(z) - zx(0) = A\hat{x}(z) + B\hat{u}(z), \qquad \hat{y}(z) = C\hat{x}(z) + D\hat{u}(z)$$

- solve for $\hat{x}(z)$

$$\hat{x}(z) = (zI - A)^{-1}zx(0) + (zI - A)^{-1}B\hat{u}(z)$$

  (note extra z in first term!)

- hence

$$\hat{y}(z) = \hat{H}(z)\hat{u}(z) + C(zI - A)^{-1}zx(0)$$

  where $\hat{H}(z) = C(zI - A)^{-1}B + D$ is the *discrete-time transfer function*

DT Example

Example: Point mass

unit point mass, with actuators applying force in directions

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} -0.5 \\ 1 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

has dynamics

$$x(k+1) = \begin{bmatrix} 1 & h & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & h \\ 0 & 0 & 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} \frac{1}{2}h^2 & 0 \\ h & 0 \\ 0 & \frac{1}{2}h^2 \\ 0 & h \end{bmatrix} \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} u_1(k) \\ u_2(k) \\ u_3(k) \end{bmatrix}$$

$$y(k) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x(k)$$

here

- $x_1, x_2 =$ position, velocity in **x**-direction
  $x_3, x_4 =$ position, velocity in **y**-direction
- $h =$ sample time; we'll use $h = 1$.
- $u_i(k)$ current applied to actuator i at time k

Example: Point mass

we would like to drive it through the positions

$$y(20) = \begin{bmatrix} 5 \\ 3 \end{bmatrix} \qquad y(40) = \begin{bmatrix} 10 \\ -1 \end{bmatrix} \qquad y(70) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

at the above times

we have

$$y(t) = \sum_{\tau=0}^{t-1} CA^{t-1-\tau} Bu(\tau) + Du(t)$$

this gives the rows of

$$\begin{bmatrix} y(20) \\ y(40) \\ y(70) \end{bmatrix} = A_{\text{act}} \begin{bmatrix} u(0) \\ \vdots \\ u(70) \end{bmatrix}$$

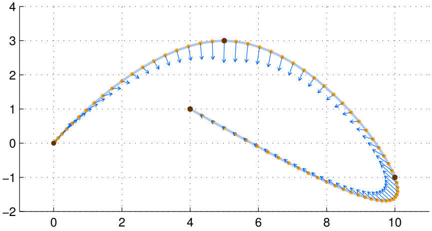
here A_{act} is 6×213.

Example: Point mass

let's find the minimum norm sequence of forces that meets the specifications

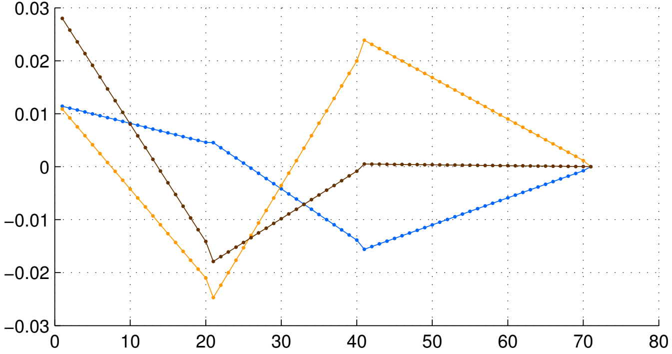
$$\begin{bmatrix} u(0) \\ \vdots \\ u(70) \end{bmatrix} = A_{\mathsf{act}}^{\dagger} \begin{bmatrix} 5 \\ 3 \\ 10 \\ -1 \\ 4 \\ 1 \end{bmatrix}$$

trajectory is



Example: Point mass

sequence of force inputs is



Coordinates and Standard Forms

Change of coordinates

start with LDS $\dot{x} = Ax + Bu$, $y = Cx + Du$

change coordinates in $\mathbb{R}^n$ to $\tilde{x}$, with $x = T\tilde{x}$

then

$$\dot{\tilde{x}} = T^{-1}\dot{x} = T^{-1}(Ax + Bu) = T^{-1}AT\tilde{x} + T^{-1}Bu$$

hence LDS can be expressed as

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u, \qquad y = \tilde{C}\tilde{x} + \tilde{D}u$$

where

$$\tilde{A} = T^{-1}AT, \qquad \tilde{B} = T^{-1}B, \qquad \tilde{C} = CT, \qquad \tilde{D} = D$$

Impulse Response is same (since u, y aren't affected):

$$\tilde{C}e^{t\tilde{A}}\tilde{B} = Ce^{tA}B$$

Standard forms for LDS

can change coordinates to put A in various forms (diagonal, real modal, Jordan $\ldots$)

$e.g.$, to put LDS in *diagonal form*, find T s.t.

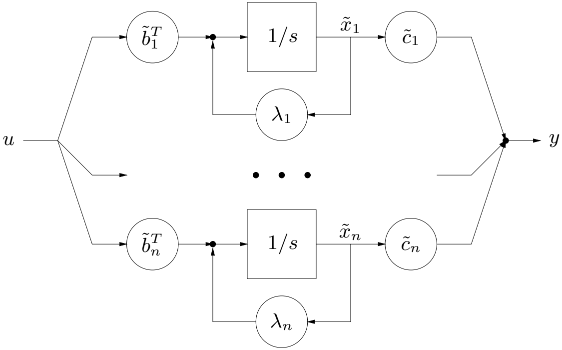
$$T^{-1}AT = \textbf{diag}(\lambda_1, \ldots, \lambda_n)$$

write

$$T^{-1}B = \begin{bmatrix} \tilde{b}_1^\mathsf{T} \\ \vdots \\ \tilde{b}_n^\mathsf{T} \end{bmatrix}, \qquad CT = \begin{bmatrix} \tilde{c}_1 & \cdots & \tilde{c}_n \end{bmatrix}$$

Diagonal form

$$\dot{\tilde{x}}_i = \lambda_i \tilde{x}_i + \tilde{b}_i^{\mathsf{T}} u, \qquad y = \sum_{i=1}^{n} \tilde{c}_i \tilde{x}_i$$

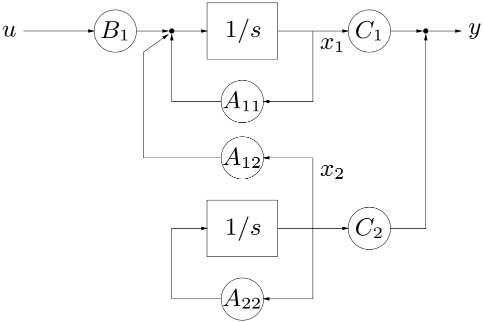


(here we assume $D = 0$)

Structure

interesting when there is structure, *e.g.*, with $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u, \qquad y = \begin{bmatrix} C_1 & C_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



- x_2 is not affected by input u, *i.e.*, x_2 propagates autonomously
- x_2 affects y directly and through x_1