Realizations and interconnections

- realizations and state-space notation
- poles, properness, SISO and MIMO
- equivalence of realizations
- operations: products, sums, inverses, feedback

Realizations and interconnections

State-space notation

Suppose

$$\hat{G}(z) = C(zI - A)^{-1}B + D$$

We define the notation

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \hat{G}$$

This block-matrix notation means the *rational function*, and we can also write

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right](z) = C(zI - A)^{-1}B + D$$

Realization

given a rational transfer function $G(s)$ (or $G(z)$ in discrete time), a *realization* of G is a tuple (A, B, C, D) such that

$$G(s) = C(sI - A)^{-1}B + D$$

the *order* (or *dimension*) of the realization is n, the size of A

- a realization is not unique: many different (A, B, C, D) can give the same $G(s)$
- different realizations can have different dimensions
- *given* (A, B, C, D), the transfer function is determined; going the other direction (from G to (A, B, C, D)) is the *realization problem*

Realization examples

example 1: $G(s) = \dfrac{1}{s-1}$

verify: $C(sI - A)^{-1}B + D = 1 \cdot \dfrac{1}{s-1} \cdot 1 + 0 = \dfrac{1}{s-1}$

realization: $A = [1]$, $B = [1]$, $C = [1]$, $D = 0$ (order 1)

example 2: $G(s) = \dfrac{3}{s+2}$

realization: $A = [-2]$, $B = [1]$, $C = [3]$, $D = 0$ (order 1)

or: $A = [-2]$, $B = [3]$, $C = [1]$, $D = 0$ (also order 1)

both are valid realizations (the factorization of the residue $3 = C \cdot B$ into C and B is not unique)

example 3: $G(s) = 5$ (constant, no dynamics)

realization: $D = [5]$ (order 0)

Higher-order examples

example 4: $G(s) = \dfrac{s+3}{(s+1)(s+2)} = \dfrac{s+3}{s^2+3s+2}$

one realization is

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 1 \end{bmatrix}, \quad D = 0$$

verify: $C(sI-A)^{-1}B = \begin{bmatrix} 3 & 1 \end{bmatrix} \dfrac{1}{s^2+3s+2} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \dfrac{s+3}{s^2+3s+2}$

example 5: non-uniqueness of dimension

$G(s) = \dfrac{1}{s+1}$ has order-1 realization: $A = [-1]$, $B = [1]$, $C = [1]$, $D = 0$

but also an order-2 realization: $A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $D = 0$

(the mode at $\lambda = -2$ does not appear in $G(s)$)

MIMO example

example 6: $A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $D = 0$

transfer function is 2×2:

$$G(s) = C(sI - A)^{-1}B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{1}{s+2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+2} \\ 0 & \frac{1}{s+2} \end{bmatrix}$$

- $G_{11}(s)$: input 1 to output 1, pole at $s = -1$
- $G_{12}(s)$: input 2 to output 1, pole at $s = -2$
- $G_{21}(s) = 0$: input 1 does not affect output 2
- $G_{22}(s)$: input 2 to output 2, pole at $s = -2$

Constructing realizations: partial fractions

given: SISO transfer function $G(s) = \dfrac{n(s)}{d(s)}$, strictly proper

expand in partial fractions (assuming distinct poles):

$$G(s) = \frac{r_1}{s-\lambda_1} + \frac{r_2}{s-\lambda_2} + \cdots + \frac{r_n}{s-\lambda_n}$$

read off the *diagonal realization*:

$$A = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} r_1 & \cdots & r_n \end{bmatrix}, \quad D = 0$$

verify: $C(sI - A)^{-1}B = \sum_i r_i \cdot \frac{1}{s-\lambda_i} \cdot 1 = G(s)$

Example: partial fraction realization

$$G(s) = \frac{s+3}{(s+1)(s+2)} = \frac{2}{s+1} + \frac{-1}{s+2}$$

diagonal realization:

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & -1 \end{bmatrix}, \quad D = 0$$

cf. the companion form realization from example 4:

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 1 \end{bmatrix}, \quad D = 0$$

these are two different realizations of the same $G(s)$; they are related by a change of coordinates (T is the eigenvector matrix of the companion A)

Constructing realizations: companion form

given: SISO $G(s) = \dfrac{b_{n-1}s^{n-1} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}$ (strictly proper)

the *companion form* realization is:

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} b_0 & b_1 & \cdots & b_{n-1} \end{bmatrix}, \quad D = 0$$

- denominator coefficients go in last row of A
- numerator coefficients go in C
- $\mathbf{det}(sI - A) = s^n + a_{n-1}s^{n-1} + \cdots + a_0$ (characteristic polynomial read from A)

Non-strictly-proper case

if $G(s)$ is proper but not strictly proper:

$$G(s) = \frac{b_n s^n + b_{n-1} s^{n-1} + \cdots + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_0}$$

perform polynomial long division:

$$G(s) = b_n + \frac{\tilde{b}_{n-1} s^{n-1} + \cdots + \tilde{b}_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_0}$$

then $D = b_n$ and apply companion form to the strictly proper remainder

example: $G(s) = \dfrac{s+3}{s+1} = 1 + \dfrac{2}{s+1}$

realization: $A = [-1]$, $B = [1]$, $C = [2]$, $D = 1$

Proper and strictly proper

$G(s) = C(sI - A)^{-1}B + D$

as $s \to \infty$: $(sI - A)^{-1} \to 0$, so $G(s) \to D$

- G is *proper* if $G(\infty)$ is finite (always true for a state-space realization)
- G is *strictly proper* if $D = 0$, $i.e.$, $G(s) \to 0$ as $s \to \infty$

strictly proper means no direct feedthrough from input to output

SISO and MIMO:

- $m = p = 1$: *single-input single-output* (SISO); $G(s)$ is a scalar rational function
- $m > 1$ or $p > 1$: *multi-input multi-output* (MIMO); $G(s) \in \mathbb{R}(s)^{p \times m}$ is a rational matrix

Poles

the *poles* of $G(s) = C(sI - A)^{-1}B + D$ are the values of s where $G(s)$ is singular ($i.e.$, blows up)

since $(sI - A)^{-1} = \text{adj}(sI - A)/\,\mathbf{det}(sI - A)$, every pole is an eigenvalue of A

- for SISO: $G(s) = n(s)/d(s)$; poles are the roots of $d(s)$ (after cancelling common factors with $n(s)$)
- some eigenvalues of A may cancel and not appear as poles of $G(s)$ (we will see later that this happens exactly when the realization is not 'minimal')

Changing coordinates

It is easy to see that if T is invertible, then

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} T^{-1}AT & T^{-1}B \\ \hline CT & D \end{array}\right]$$

We call two realizations A_1, B_1, C_1, D_1 and A_2, B_2, C_2, D_2 *equivalent* if they have the same transfer function

$$\left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right] = \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right]$$

equivalently (assuming $D_1 = D_2$), any of:

- same impulse response: $C_1 e^{A_1 t} B_1 = C_2 e^{A_2 t} B_2$ for all $t \geq 0$
- same *Markov parameters*: $C_1 A_1^k B_1 = C_2 A_2^k B_2$ for $k = 0, 1, 2, \ldots$

note: equivalent realizations can have different dimensions, different eigenvalues, and different state trajectories — they simply have the same input-output behavior

Products

Suppose

$$G_1 = \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right] \qquad G_2 = \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right]$$

Two realizations for the product are

$$G_1 G_2 = \left[\begin{array}{cc|c} A_1 & B_1 C_2 & B_1 D_2 \\ 0 & A_2 & B_2 \\ \hline C_1 & D_1 C_2 & D_1 D_2 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} A_2 & 0 & B_2 \\ B_1 C_2 & A_1 & B_1 D_2 \\ \hline D_1 C_2 & C_1 & D_1 D_2 \end{array}\right]$$

To prove this, use the fact that

$$\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & -A^{-1} B C^{-1} \\ 0 & C^{-1} \end{bmatrix}$$

Sums

For the sum we have

$$G_1 + G_2 = \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ \hline C_1 & C_2 & D_1 + D_2 \end{array}\right]$$

Inverses

We can also invert a realization. If D is invertible, then

$$G^{-1} = \left[\begin{array}{c|c} A - BD^{-1}C & -BD^{-1} \\ \hline D^{-1}C & D^{-1} \end{array}\right]$$

this means

$$\left[\begin{array}{c|c} A - BD^{-1}C & -BD^{-1} \\ \hline D^{-1}C & D^{-1} \end{array}\right] \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = I$$

where we multiply the two rational functions and cancel all common factors.

Left Inverses

Similarly, if D is left invertible, then

$$\left[\begin{array}{c|c} A - BD^{\dagger}C & -BD^{\dagger} \\ \hline D^{\dagger}C & D^{\dagger} \end{array}\right] \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = I$$

Stacking

$$\begin{bmatrix} G_1 & G_2 \end{bmatrix} = \left[\begin{array}{cc|cc} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ \hline C_1 & C_2 & D_1 & D_2 \end{array}\right]$$

and

$$\begin{bmatrix} G_1 \\ G_2 \end{bmatrix} = \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ \hline C_1 & 0 & D_1 \\ 0 & C_2 & D_2 \end{array}\right]$$

Feedback interconnection

plant G with controller K in negative feedback: $u = r - Kw$, $w = Gu$

the closed-loop from r to w is $(I + GK)^{-1}G$

for $G = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$ and static feedback K:

$$(I + GK)^{-1}G = \left[\begin{array}{c|c} A - BKC & B \\ \hline C & 0 \end{array}\right]$$

for dynamic controller $K = \left[\begin{array}{c|c} A_k & B_k \\ \hline C_k & D_k \end{array}\right]$:

$$(I + GK)^{-1}G = \left[\begin{array}{cc|c} A - BD_kC & -BC_k & B \\ B_kC & A_k & 0 \\ \hline C & 0 & 0 \end{array}\right]$$

(cf. separation principle: the augmented system with plant and observer-based controller)

well-posedness: requires $I + D_kD$ invertible; automatic when either system is strictly proper

Dimension of composed systems

every operation increases the state dimension (or leaves it unchanged):

$G_1 + G_2$	dimension $n_1 + n_2$
$G_1 G_2$	dimension $n_1 + n_2$
$\begin{bmatrix} G_1 & G_2 \end{bmatrix}$, $\begin{bmatrix} G_1 \\ G_2 \end{bmatrix}$	dimension $n_1 + n_2$
G^{-1}	dimension n
feedback	dimension $n + n_k$

but these operations can result in realizations that are not the smallest possible

example: $G_1 + G_2$ has dimension $n_1 + n_2$, but if $G_1 = G_2$ then $G_1 + G_2 = 2G_1$ has a simple realization with dimension n_1