

## Controllability and state transfer

- ▶ state transfer and reachability
- ▶ invariance of reachable subspace
- ▶ minimum norm inputs
- ▶ continuous-time controllability
- ▶ continuous-time minimum norm inputs
- ▶ reachability Gramian

# State transfer and reachability

## State transfer

consider  $\dot{x} = Ax + Bu$  (or  $x(t+1) = Ax(t) + Bu(t)$ ) over time interval  $[t_i, t_f]$

we say input  $u : [t_i, t_f] \rightarrow \mathbb{R}^m$  *steers* or *transfers* state from  $x(t_i)$  to  $x(t_f)$  (over time interval  $[t_i, t_f]$ )

(subscripts stand for *initial* and *final*)

questions:

- ▶ where can  $x(t_i)$  be transferred to at  $t = t_f$ ?
- ▶ how quickly can  $x(t_i)$  be transferred to some  $x_{\text{target}}$ ?
- ▶ how do we find a  $u$  that transfers  $x(t_i)$  to  $x(t_f)$ ?
- ▶ how do we find a 'small' or 'efficient'  $u$  that transfers  $x(t_i)$  to  $x(t_f)$ ?

## Reachability

consider state transfer from  $x(0) = 0$  to  $x(t)$

we say  $x(t)$  is *reachable* (in  $t$  seconds or epochs)

we define  $\mathcal{R}_t \subseteq \mathbb{R}^n$  as the set of points reachable in  $t$  seconds or epochs

for CT system  $\dot{x} = Ax + Bu$ ,

$$\mathcal{R}_t = \left\{ \int_0^t e^{(t-\tau)A} B u(\tau) d\tau \mid u : [0, t] \rightarrow \mathbb{R}^m \right\}$$

and for DT system  $x(t+1) = Ax(t) + Bu(t)$ ,

$$\mathcal{R}_t = \left\{ \sum_{\tau=0}^{t-1} A^{t-1-\tau} B u(\tau) \mid u(0), \dots, u(t-1) \in \mathbb{R}^m \right\}$$

## Reachable set

- ▶  $\mathcal{R}_t$  is a subspace of  $\mathbb{R}^n$
- ▶  $\mathcal{R}_t \subseteq \mathcal{R}_s$  if  $t \leq s$   
(*i.e.*, can reach more points given more time)

we define the *reachable set*  $\mathcal{R}$  as the set of points reachable for some  $t$ :

$$\mathcal{R} = \bigcup_{t \geq 0} \mathcal{R}_t$$

## Reachability for discrete-time LDS

DT system  $x(t+1) = Ax(t) + Bu(t)$ ,  $x(t) \in \mathbb{R}^n$

$$x(t) = C_t \begin{bmatrix} u(t-1) \\ \vdots \\ u(0) \end{bmatrix}$$

where  $C_t = [B \quad AB \quad \dots \quad A^{t-1}B]$  so reachable set at  $t$  is  $\mathcal{R}_t = \text{range}(C_t)$

by Cayley-Hamilton theorem, we can express each  $A^k$  for  $k \geq n$  as linear combination of  $A^0, \dots, A^{n-1}$

hence for  $t \geq n$ ,  $\text{range}(C_t) = \text{range}(C_n)$

thus we have

$$\mathcal{R}_t = \begin{cases} \text{range}(C_t) & t < n \\ \text{range}(C) & t \geq n \end{cases}$$

where  $C = C_n$  is called the *controllability matrix*

- ▶ any state that can be reached can be reached by  $t = n$
- ▶ the reachable set is  $\mathcal{R} = \text{range}(C)$

## Controllable system

system is called *reachable* or *controllable* if all states are reachable (*i.e.*,  $\mathcal{R} = \mathbb{R}^n$ )

system is reachable if and only if  $\text{rank}(C) = n$

**example:**  $x(t+1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$

controllability matrix is  $C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

hence system is not controllable; reachable set is

$$\mathcal{R} = \text{range}(C) = \{ x \mid x_1 = x_2 \}$$

## Invariance of reachable subspace

**fact:** the reachable subspace  $\mathcal{R} = \mathbf{range}(C)$  is invariant, *i.e.*, if  $z \in \mathbf{range}(C)$ , then  $Az \in \mathbf{range}(C)$

**proof:** suppose  $z \in \mathbf{range}(C)$ , *i.e.*,  $z = \sum_{i=0}^{n-1} A^i B u(i)$

then

$$Az = \sum_{i=0}^{n-1} A^{i+1} B u(i) = \sum_{i=0}^{n-2} A^{i+1} B u(i) + A^n B u(n-1)$$

by C-H,  $A^n = -\alpha_0 I - \dots - \alpha_{n-1} A^{n-1}$  where  $\det(sI - A) = s^n + \alpha_{n-1} s^{n-1} + \dots + \alpha_0$

so  $A^n B u(n-1) \in \mathbf{range}(C)$ , and hence  $Az \in \mathbf{range}(C)$

## General state transfer

with  $t_f > t_i$ ,

$$x(t_f) = A^{t_f - t_i} x(t_i) + C_{t_f - t_i} \begin{bmatrix} u(t_f - 1) \\ \vdots \\ u(t_i) \end{bmatrix}$$

hence can transfer  $x(t_i)$  to  $x(t_f) = x_{\text{des}}$

$$\Leftrightarrow x_{\text{des}} - A^{t_f - t_i} x(t_i) \in \mathcal{R}_{t_f - t_i}$$

- ▶ general state transfer reduces to reachability problem
- ▶ if system is controllable any state transfer can be achieved in  $\leq n$  steps
- ▶ important special case: driving state to zero (sometimes called regulating or controlling state)

## Minimum norm inputs

## Least-norm input for reachability

assume system is reachable,  $\text{rank}(C_t) = n$

to steer  $x(0) = 0$  to  $x(t) = x_{\text{des}}$ , inputs  $u(0), \dots, u(t-1)$  must satisfy

$$x_{\text{des}} = C_t \begin{bmatrix} u(t-1) \\ \vdots \\ u(0) \end{bmatrix}$$

among all  $u$  that steer  $x(0) = 0$  to  $x(t) = x_{\text{des}}$ , the one that minimizes  $\sum_{\tau=0}^{t-1} \|u(\tau)\|^2$  is given by

$$\begin{bmatrix} u_{\text{ln}}(t-1) \\ \vdots \\ u_{\text{ln}}(0) \end{bmatrix} = C_t^T (C_t C_t^T)^{-1} x_{\text{des}}$$

$u_{\text{ln}}$  is called *least-norm* or *minimum energy* input that effects state transfer

can express as

$$u_{\text{ln}}(\tau) = B^T (A^T)^{(t-1-\tau)} \left( \sum_{s=0}^{t-1} A^s B B^T (A^T)^s \right)^{-1} x_{\text{des}},$$

for  $\tau = 0, \dots, t-1$

## Minimum energy

$\mathcal{E}_{\min}$ , the minimum value of  $\sum_{\tau=0}^{t-1} \|u(\tau)\|^2$  required to reach  $x(t) = x_{\text{des}}$ , is sometimes called *minimum energy* required to reach  $x(t) = x_{\text{des}}$

$$\begin{aligned}\mathcal{E}_{\min} &= \sum_{\tau=0}^{t-1} \|u_{\text{in}}(\tau)\|^2 = (C_t^\top (C_t C_t^\top)^{-1} x_{\text{des}})^\top C_t^\top (C_t C_t^\top)^{-1} x_{\text{des}} \\ &= x_{\text{des}}^\top (C_t C_t^\top)^{-1} x_{\text{des}} \\ &= x_{\text{des}}^\top \left( \sum_{\tau=0}^{t-1} A^\tau B B^\top (A^\top)^\tau \right)^{-1} x_{\text{des}}\end{aligned}$$

- ▶  $\mathcal{E}_{\min}(x_{\text{des}}, t)$  gives measure of how hard it is to reach  $x(t) = x_{\text{des}}$  from  $x(0) = 0$  (i.e., how large a  $u$  is required)
- ▶  $\mathcal{E}_{\min}(x_{\text{des}}, t)$  gives practical measure of controllability/reachability (as function of  $x_{\text{des}}, t$ )
- ▶ ellipsoid  $\{ z \mid \mathcal{E}_{\min}(z, t) \leq 1 \}$  shows points in state space reachable at  $t$  with one unit of energy (shows directions that can be reached with small inputs, and directions that can be reached only with large inputs)

## Energy dependence on time

$\mathcal{E}_{\min}$  as function of  $t$ :

if  $t \geq s$  then

$$\sum_{\tau=0}^{t-1} A^\tau B B^\top (A^\top)^\tau \geq \sum_{\tau=0}^{s-1} A^\tau B B^\top (A^\top)^\tau$$

hence

$$\left( \sum_{\tau=0}^{t-1} A^\tau B B^\top (A^\top)^\tau \right)^{-1} \leq \left( \sum_{\tau=0}^{s-1} A^\tau B B^\top (A^\top)^\tau \right)^{-1}$$

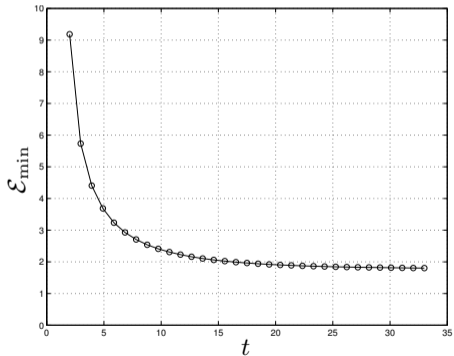
so  $\mathcal{E}_{\min}(x_{\text{des}}, t) \leq \mathcal{E}_{\min}(x_{\text{des}}, s)$

*i.e.*: takes less energy to get somewhere more leisurely

## Example:

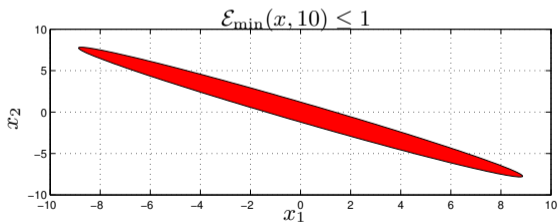
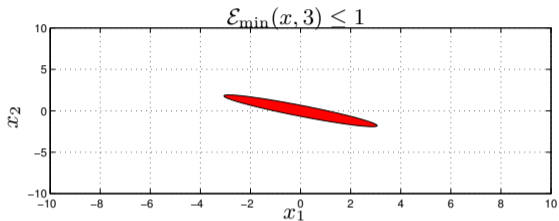
$$x(t+1) = \begin{bmatrix} 1.75 & 0.8 \\ -0.95 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

$\mathcal{E}_{\min}(z, t)$  for  $z = [1 \ 1]^T$ :



## Example

ellipsoids  $\mathcal{E}_{\min} \leq 1$  for  $t = 3$  and  $t = 10$ :



## Discrete-time reachability Gramian

suppose  $x(t+1) = Ax(t) + Bu(t)$  is controllable and stable

then  $\sum_{\tau=0}^{t-1} A^\tau BB^\top (A^\top)^\tau$  converges as  $t \rightarrow \infty$  since  $A^\tau$  decays geometrically

the matrix

$$W_r = \sum_{\tau=0}^{\infty} A^\tau BB^\top (A^\top)^\tau$$

is called the *reachability* (or *controllability*) *Gramian*

$W_r$  satisfies the matrix equation

$$W_r - AW_r A^\top = BB^\top$$

which is called the controllability *Lyapunov equation* (a linear equation in  $W_r$  which can be efficiently solved)

# Continuous-time controllability

## Continuous-time reachability

consider now  $\dot{x} = Ax + Bu$  with  $x(t) \in \mathbb{R}^n$

reachable set at time  $t$  is

$$\mathcal{R}_t = \left\{ \int_0^t e^{(t-\tau)A} B u(\tau) d\tau \mid u : [0, t] \rightarrow \mathbb{R}^m \right\}$$

**fact:** for  $t > 0$ ,  $\mathcal{R}_t = \mathcal{R} = \text{range}(C)$ , where

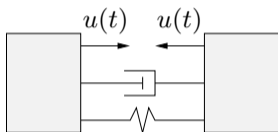
$$C = [B \quad AB \quad \dots \quad A^{n-1}B]$$

is the controllability matrix of  $(A, B)$

- ▶ same  $\mathcal{R}$  as discrete-time system
- ▶ for continuous-time system, any reachable point can be reached as fast as you like (with large enough  $u$ )

## Example

- ▶ unit masses at  $y_1, y_2$ , connected by unit springs, dampers
- ▶ input is tension between masses
- ▶ state is  $x = [y^T \dot{y}^T]^T$



system is

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} u$$

- ▶ can we maneuver state anywhere, starting from  $x(0) = 0$ ?
- ▶ if not, where can we maneuver state?

## Example

controllability matrix is

$$C = [B \quad AB \quad A^2B \quad A^3B] = \begin{bmatrix} 0 & 1 & -2 & 2 \\ 0 & -1 & 2 & -2 \\ 1 & -2 & 2 & 0 \\ -1 & 2 & -2 & 0 \end{bmatrix}$$

hence reachable set is

$$\mathcal{R} = \text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

we can reach states with  $y_1 = -y_2$ ,  $\dot{y}_1 = -\dot{y}_2$ , *i.e.*, precisely the differential motions  
it's obvious — internal force does not affect center of mass position or total momentum!

## Continuous-time minimum norm inputs

## Least-norm input for reachability

(also called *minimum energy input*)

assume that  $\dot{x} = Ax + Bu$  is reachable

we seek  $u$  that steers  $x(0) = 0$  to  $x(t) = x_{\text{des}}$  and minimizes

$$\int_0^t \|u(\tau)\|^2 d\tau$$

let's discretize system with interval  $h = t/N$

(we'll let  $N \rightarrow \infty$  later)

thus  $u$  is piecewise constant:

$$u(\tau) = u_d(k) \text{ for } kh \leq \tau < (k+1)h, \quad k = 0, \dots, N-1$$

## Least-norm input for reachability

so

$$x(t) = \begin{bmatrix} B_d & A_d B_d & \cdots & A_d^{N-1} B_d \end{bmatrix} \begin{bmatrix} u_d(N-1) \\ \vdots \\ u_d(0) \end{bmatrix}$$

where

$$A_d = e^{hA}, \quad B_d = \int_0^h e^{\tau A} d\tau B$$

least-norm  $u_d$  that yields  $x(t) = x_{\text{des}}$  is

$$u_{\text{dln}}(k) = B_d^T (A_d^T)^{(N-1-k)} \left( \sum_{i=0}^{N-1} A_d^i B_d B_d^T (A_d^T)^i \right)^{-1} x_{\text{des}}$$

let's express in terms of  $A$ :

$$B_d^T (A_d^T)^{(N-1-k)} = B_d^T e^{(t-\tau)A^T}$$

where  $\tau = t(k+1)/N$

## Least-norm input for reachability

for  $N$  large,  $B_d \approx (t/N)B$ , so this is approximately

$$(t/N)B^T e^{(t-\tau)A^T}$$

similarly

$$\sum_{i=0}^{N-1} A_d^i B_d B_d^T (A_d^T)^i = \sum_{i=0}^{N-1} e^{(ti/N)A} B_d B_d^T e^{(ti/N)A^T} \approx (t/N) \int_0^t e^{\bar{t}A} B B^T e^{\bar{t}A^T} d\bar{t}$$

for large  $N$

hence least-norm discretized input is approximately

$$u_{\text{ln}}(\tau) = B^T e^{(t-\tau)A^T} \left( \int_0^t e^{\bar{t}A} B B^T e^{\bar{t}A^T} d\bar{t} \right)^{-1} x_{\text{des}}, \quad 0 \leq \tau \leq t$$

for large  $N$

hence, this is the least-norm continuous input

- ▶ can make  $t$  small, but get larger  $u$
- ▶ cf. DT solution: sum becomes integral

## Minimum energy

min energy is

$$\int_0^t \|u_{\text{in}}(\tau)\|^2 d\tau = x_{\text{des}}^T W_r(t)^{-1} x_{\text{des}}$$

where

$$W_r(t) = \int_0^t e^{\tau A} B B^T e^{\tau A^T} d\tau$$

can show

$$\begin{aligned} (A, B) \text{ controllable} &\Leftrightarrow W_r(t) > 0 \text{ for all } t > 0 \\ &\Leftrightarrow W_r(s) > 0 \text{ for some } s > 0 \end{aligned}$$

in fact,  $\text{range}(W_r(t)) = \mathcal{R}$  for any  $t > 0$

## Minimum energy over infinite horizon

the matrix

$$P = \lim_{t \rightarrow \infty} \left( \int_0^t e^{\tau A} B B^T e^{\tau A^T} d\tau \right)^{-1}$$

always exists, and gives minimum energy required to reach a point  $x_{\text{des}}$  (with no limit on  $t$ ):

$$\min \left\{ \int_0^t \|u(\tau)\|^2 d\tau \mid x(0) = 0, x(t) = x_{\text{des}} \right\} = x_{\text{des}}^T P x_{\text{des}}$$

- ▶ if  $A$  is stable,  $P > 0$  (*i.e.*, can't get anywhere for free)
- ▶ if  $A$  is not stable, then  $P$  can have nonzero nullspace
- ▶  $Pz = 0, z \neq 0$  means can get to  $z$  using  $u$ 's with energy as small as you like ( $u$  just gives a little kick to the state; the instability carries it out to  $z$  efficiently)

## Reachability Gramian

if  $\dot{x} = Ax + Bu$  is controllable and stable

then  $W_r(t)$  converges as  $t \rightarrow \infty$  to

$$W_r = \int_0^{\infty} e^{A\bar{t}} BB^T e^{A^T\bar{t}} d\bar{t},$$

the *reachability* (or *controllability*) *Gramian*. The CT reachability Gramian  $W_r$  satisfies the matrix equation

$$AW_r + W_r A^T + BB^T = 0$$

which is called the controllability *Lyapunov equation* to see this, note that

$$\frac{d}{dt} e^{tA} BB^T e^{A^T t} = A e^{tA} BB^T e^{A^T t} + e^{tA} BB^T e^{A^T t} A^T$$

integrate from  $t = 0$  to  $\infty$  to get:

$$e^{tA} BB^T e^{A^T t} \Big|_0^{\infty} = AW_r + W_r A^T$$

which gives the Lyapunov equation (a linear equation in  $W_r$  which can be efficiently solved)

## General state transfer

consider state transfer from  $x(t_i)$  to  $x(t_f) = x_{\text{des}}$ ,  $t_f > t_i$

since

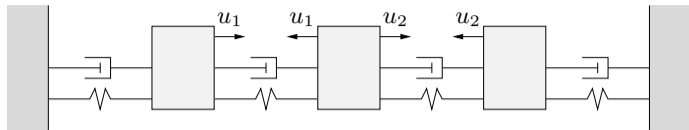
$$x(t_f) = e^{(t_f - t_i)A} x(t_i) + \int_{t_i}^{t_f} e^{(t_f - \tau)A} B u(\tau) d\tau$$

$u$  steers  $x(t_i)$  to  $x(t_f) = x_{\text{des}} \Leftrightarrow$

$u$  (shifted by  $t_i$ ) steers  $x(0) = 0$  to  $x(t_f - t_i) = x_{\text{des}} - e^{(t_f - t_i)A} x(t_i)$

- ▶ general state transfer reduces to reachability problem
- ▶ if system is controllable, any state transfer can be effected
  - ▶ in 'zero' time with impulsive inputs
  - ▶ in any positive time with non-impulsive inputs

## Example



- ▶ unit masses, springs, dampers
- ▶  $u_1$  is force between 1st & 2nd masses
- ▶  $u_2$  is force between 2nd & 3rd masses
- ▶  $y \in \mathbb{R}^3$  is displacement of masses 1,2,3, state  $x = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$

system is:

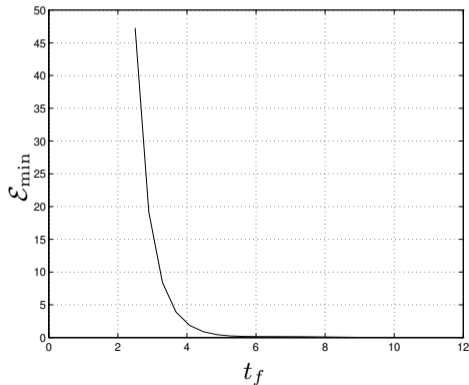
$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & -2 & 1 & 0 \\ 1 & -2 & 1 & 1 & -2 & 1 \\ 0 & 1 & -2 & 0 & 1 & -2 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

## Example

steer state from  $x(0) = e_1$  to  $x(t_f) = 0$

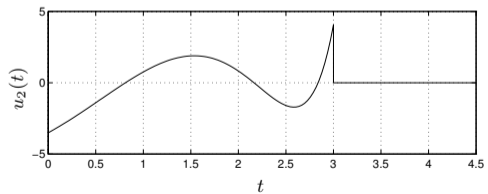
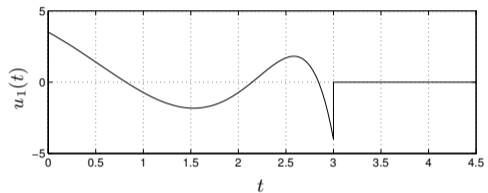
*i.e.*, control initial state  $e_1$  to zero at  $t = t_f$

$$\mathcal{E}_{\min} = \int_0^{t_f} \|u_{\min}(\tau)\|^2 d\tau \text{ vs. } t_f:$$



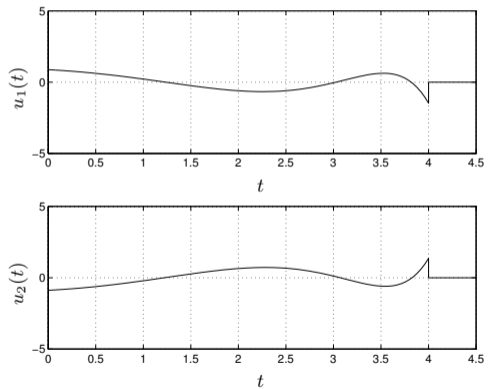
## Example

for  $t_f = 3$ ,  $u = u_{1n}$  is:



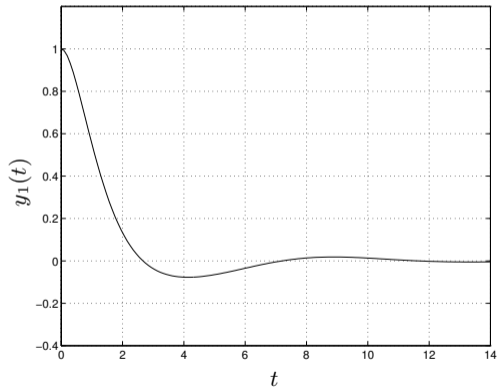
## Example

and for  $t_f = 4$ :



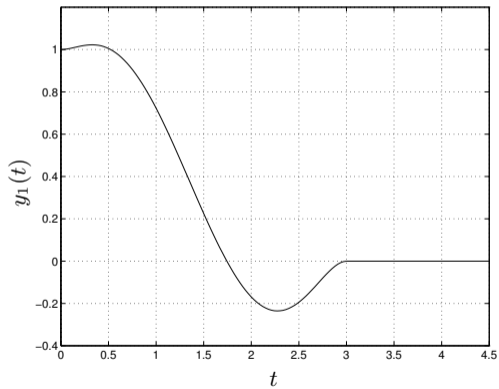
## Example

output  $y_1$  for  $u = 0$ :



## Example

output  $y_1$  for  $u = u_{in}$  with  $t_f = 3$ :



## Example

output  $y_1$  for  $u = u_{in}$  with  $t_f = 4$ :

