Observability and state estimation

- state estimation
- discrete-time observability
- observers and least-squares estimation
- current state estimation
- continuous-time observability

State estimation

State estimation set up

we consider the discrete-time system

$$x(t+1) = Ax(t) + Bu(t) + w(t), \quad y(t) = Cx(t) + Du(t) + v(t)$$

- w is state *disturbance* or *noise*
- v is sensor *noise* or *error*
- A, B, C, and D are known
- u and y are observed over time interval $[0, t-1]$
- w and v are not known, but can be described statistically, or assumed small ($e.g.$, in RMS value)

State estimation problem

state estimation problem: estimate $x(s)$ from

$$u(0), \ldots, u(t-1),\ \ y(0), \ldots, y(t-1)$$

- $s = 0$: estimate initial state
- $s = t-1$: estimate current state
- $s = t$: estimate ($i.e.$, predict) next state

an algorithm or system that yields an estimate $\hat{x}(s)$ is called an *observer* or *state estimator* $\hat{x}(s)$ is denoted $\hat{x}(s|t-1)$ to show what information estimate is based on (read, "$\hat{x}(s)$ given $t-1$")

Removing effect of input

suppose

$$x(t+1) = Ax(t) + Bu(t) + w(t), \quad y(t) = Cx(t) + Du(t) + v(t)$$

define

$$x_u(t) = \sum_{\tau=0}^{t-1} A^{t-\tau-1} Bu(\tau), \quad y_u(t) = \sum_{\tau=0}^{t-1} CA^{t-\tau-1} Bu(\tau) + Du(t)$$

- these are known since A, B, C, D, and u are known
- x_u is part of state due to input
- y_u is part of output due to input

$x - x_u$ and $y - y_u$ satisfy autonomous equations

$$(x - x_u)(t+1) = A(x - x_u)(t) + w(t), \quad (y - y_u)(t) = C(x - x_u)(t) + v(t)$$

Removing effect of input

we can estimate x by

- estimating $x - x_u$
  (which satisfies autonomous equation)

- adding back term due to input, $i.e.$, x_u

so we henceforth consider autonomous case only

Discrete-time observability

Noiseless case

let's look at finding $x(0)$, with no state or measurement noise:

$$x(t+1) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t)$$

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$, then we have

$$\begin{bmatrix} y(0) \\ \vdots \\ y(t-1) \end{bmatrix} = \mathcal{O}_t x(0) + \mathcal{T}_t \begin{bmatrix} u(0) \\ \vdots \\ u(t-1) \end{bmatrix}$$

where

$$\mathcal{O}_t = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{t-1} \end{bmatrix}, \quad \mathcal{T}_t = \begin{bmatrix} D & 0 & \cdots & & \\ CB & D & 0 & \cdots & \\ \vdots & & & & \\ CA^{t-2}B & CA^{t-3}B & \cdots & CB & D \end{bmatrix}$$

- $\mathcal{O}_t$ maps initials state into resulting output over $[0, t-1]$
- $\mathcal{T}_t$ maps input to output over $[0, t-1]$

Noiseless case

hence we have

$$\mathcal{O}_t x(0) = \begin{bmatrix} y(0) \\ \vdots \\ y(t-1) \end{bmatrix} - \mathcal{T}_t \begin{bmatrix} u(0) \\ \vdots \\ u(t-1) \end{bmatrix}$$

RHS is known, $x(0)$ is to be determined

Observation

hence:

- can uniquely determine $x(0)$ from $y(0), \ldots, y(t-1)$ if and only if **null**$(\mathcal{O}_t) = \{0\}$
- **null**$(\mathcal{O}_t)$ gives ambiguity in determining $x(0)$ from $y(0), \ldots, y(t-1)$
- if $x(0) \in$ **null**$(\mathcal{O}_t)$ and $u = 0$, output is zero over interval $[0, t-1]$
- input u does not affect ability to determine $x(0)$;
  its effect can be subtracted out

Observability matrix

by C-H theorem, each A^k is linear combination of $A^0, \ldots, A^{n-1}$

hence for $t \geq n$, **null**$(\mathcal{O}_t) =$ **null**$(\mathcal{O})$ where

$$\mathcal{O} = \mathcal{O}_n = \left[\begin{array}{c} C \\ CA \\ \vdots \\ CA^{n-1} \end{array}\right]$$

is called the *observability matrix*

if $x(0)$ can be deduced from u and y over $[0, t-1]$ for any t, then $x(0)$ can be deduced from u and y over $[0, n-1]$

null$(\mathcal{O})$ is called *unobservable subspace*; describes ambiguity in determining state from input and output

system is called *observable* if **null**$(\mathcal{O}) = \{0\}$, $i.e.$, **rank**$(\mathcal{O}) = n$

Invariance of unobservable subspace

fact: the unobservable subspace **null**$(\mathcal{O})$ is invariant, *i.e.*, if $z \in \textbf{null}(\mathcal{O})$, then $Az \in \textbf{null}(\mathcal{O})$

proof: suppose $z \in \textbf{null}(\mathcal{O})$, *i.e.*, $CA^k z = 0$ for $k = 0, \ldots, n-1$

evidently $CA^k(Az) = 0$ for $k = 0, \ldots, n-2$;

$$CA^{n-1}(Az) = CA^n z = -\sum_{i=0}^{n-1} \alpha_i CA^i z = 0$$

(by C-H) where

$$\textbf{det}(sI - A) = s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_0$$

Observers for noiseless case

suppose **rank**$(\mathcal{O}_t) = n$ ($i.e.$, system is observable) and let F be any left inverse of $\mathcal{O}_t$, $i.e.$, $F\mathcal{O}_t = I$

then we have the observer

$$x(0) = F\left(\begin{bmatrix} y(0) \\ \vdots \\ y(t-1) \end{bmatrix} - \mathcal{T}_t \begin{bmatrix} u(0) \\ \vdots \\ u(t-1) \end{bmatrix} \right)$$

which deduces $x(0)$ (exactly) from u, y over $[0, t-1]$

in fact we have

$$x(\tau - t + 1) = F\left(\begin{bmatrix} y(\tau - t + 1) \\ \vdots \\ y(\tau) \end{bmatrix} - \mathcal{T}_t \begin{bmatrix} u(\tau - t + 1) \\ \vdots \\ u(\tau) \end{bmatrix} \right)$$

$i.e.$, our observer estimates what state was $t-1$ epochs ago, given past $t-1$ inputs & outputs

observer is (multi-input, multi-output) *finite impulse response* (FIR) filter, with inputs u and y, and output $\hat{x}$

Least-squares estimation

Least-squares observers

discrete-time system, with sensor noise:

$$x(t+1) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t) + v(t)$$

we assume **rank**$(\mathcal{O}_t) = n$ (hence, system is observable)

least-squares observer uses pseudo-inverse:

$$\hat{x}(0) = \mathcal{O}_t^\dagger \left(\begin{bmatrix} y(0) \\ \vdots \\ y(t-1) \end{bmatrix} - \mathcal{T}_t \begin{bmatrix} u(0) \\ \vdots \\ u(t-1) \end{bmatrix} \right)$$

where $\mathcal{O}_t^\dagger = \left(\mathcal{O}_t^\mathsf{T} \mathcal{O}_t\right)^{-1} \mathcal{O}_t^\mathsf{T}$

interpretation: $\hat{x}_{\mathrm{ls}}(0)$ minimizes discrepancy between

- output $\hat{y}$ that *would be* observed, with input u and initial state $x(0)$ (and no sensor noise), and
- output y that *was* observed,

measured as $\sum_{\tau=0}^{t-1} \|\hat{y}(\tau) - y(\tau)\|^2$

can express least-squares initial state estimate as

$$\hat{x}_{\mathrm{ls}}(0) = \left(\sum_{\tau=0}^{t-1} (A^{\mathsf{T}})^{\tau} C^{\mathsf{T}} C A^{\tau} \right)^{-1} \sum_{\tau=0}^{t-1} (A^{\mathsf{T}})^{\tau} C^{\mathsf{T}} \tilde{y}(\tau)$$

where $\tilde{y}$ is observed output with portion due to input subtracted: $\tilde{y} = y - h * u$ where h is impulse response

Least-squares observer uncertainty ellipsoid

since $\mathcal{O}_t^\dagger \mathcal{O}_t = I$, we have

$$\tilde{x}(0) = \hat{x}_{\rm ls}(0) - x(0) = \mathcal{O}_t^\dagger \begin{bmatrix} v(0) \\ \vdots \\ v(t-1) \end{bmatrix}$$

where $\tilde{x}(0)$ is the estimation error of the initial state

in particular, $\hat{x}_{\rm ls}(0) = x(0)$ if sensor noise is zero
($i.e.$, observer recovers exact state in noiseless case)

now assume sensor noise is unknown, but has RMS value $\leq \alpha$,

$$\frac{1}{t} \sum_{\tau=0}^{t-1} \|v(\tau)\|^2 \leq \alpha^2$$

set of possible estimation errors is ellipsoid

$$\tilde{x}(0) \in \mathcal{E}_{\mathtt{unc}} = \left\{ \mathcal{O}_t^\dagger \begin{bmatrix} v(0) \\ \vdots \\ v(t-1) \end{bmatrix} \;\middle|\; \frac{1}{t}\sum_{\tau=0}^{t-1} \|v(\tau)\|^2 \leq \alpha^2 \right\}$$

$\mathcal{E}_{\mathtt{unc}}$ is ‘uncertainty ellipsoid’ for $x(0)$ (least-square gives best $\mathcal{E}_{\mathtt{unc}}$)

shape of uncertainty ellipsoid determined by matrix

$$\left(\mathcal{O}_t^\mathsf{T}\mathcal{O}_t\right)^{-1} = \left(\sum_{\tau=0}^{t-1} (A^\mathsf{T})^\tau C^\mathsf{T} C A^\tau\right)^{-1}$$

maximum norm of error is

$$\|\hat{x}_{\mathrm{ls}}(0) - x(0)\| \leq \alpha\sqrt{t}\|\mathcal{O}_t^\dagger\|$$

Statistical interpretation of least-squares observer

suppose sensor noise is IID $\mathcal{N}(0, \sigma I)$

- called *white noise*
- each sensor has noise variance σ

then $\hat{x}_{\mathrm{ls}}(0)$ is MMSE estimate of $x(0)$ when $x(0)$ is deterministic (or has 'infinite' prior variance)

estimation error $z = \hat{x}_{\mathrm{ls}}(0) - x(0)$ can be expressed as

$$z = \mathcal{O}_t^\dagger \begin{bmatrix} v(0) \\ \vdots \\ v(t-1) \end{bmatrix}$$

hence $z \sim \mathcal{N}\left(0, \sigma \mathcal{O}^\dagger \mathcal{O}^{\dagger T}\right)$

Statistical interpretation of least-squares observer

$i.e.$, covariance of least-squares initial state estimation error is

$$\sigma \mathcal{O}^{\dagger} \mathcal{O}^{\dagger T} = \sigma \left(\sum_{\tau=0}^{t-1} (A^{\mathsf{T}})^{\tau} C^{\mathsf{T}} C A^{\tau} \right)^{-1}$$

(cf. min energy required to reach a state)

we'll assume $\sigma = 1$ to simplify

matrix $\left(\sum_{\tau=0}^{t-1} (A^{\mathsf{T}})^{\tau} C^{\mathsf{T}} C A^{\tau} \right)^{-1}$ gives measure of 'how observable' the state is, over $[0, t-1]$

Infinite horizon uncertainty ellipsoid

the matrix

$$P = \lim_{t\to\infty} \left(\sum_{\tau=0}^{t-1} (A^\mathsf{T})^\tau C^\mathsf{T} C A^\tau \right)^{-1}$$

always exists, and gives the limiting uncertainty in estimating $x(0)$ from u, y over longer and longer periods:

$$\lim_{t\to\infty} \mathbf{E}(\hat{x}_{\mathrm{ls}}(0|t-1) - x(0))(\hat{x}_{\mathrm{ls}}(0|t-1) - x(0))^\mathsf{T} = P$$

- if A is stable, $P > 0$
  $i.e.$, can't estimate initial state perfectly even with infinite number of measurements $u(t)$, $y(t)$, $t = 0, \ldots$ (since memory of $x(0)$ fades . . .)
- if A is not stable, then P can have nonzero nullspace
  $i.e.$, initial state estimation error gets arbitrarily small (at least in some directions) as more and more of signals u and y are observed

Observability Gramian

suppose system

$$x(t+1) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t)$$

is observable and stable

then $\sum_{\tau=0}^{t-1} (A^T)^\tau C^T C A^\tau$ converges as $t \to \infty$ since A^τ decays geometrically

the matrix $W_o = \sum_{\tau=0}^{\infty} (A^T)^\tau C^T C A^\tau$ is called the *observability Gramian* W_o satisfies the matrix equation

$$W_o - A^T W_o A = C^T C$$

which is called the observability *Lyapunov equation* (and can be solved exactly and efficiently)

Current state estimation

Current state estimation

we have concentrated on estimating $x(0)$ from

$$u(0), \ldots, u(t-1), \; y(0), \ldots, y(t-1)$$

now we look at estimating $x(t-1)$ from this data

we assume

$$x(t+1) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t) + v(t)$$

- no state noise
- $D = 0$ (just to simplify)
- v is white, $i.e.$, IID $\mathcal{N}(0, \sigma I)$

Current state estimation

using

$$x(t-1) = A^{t-1}x(0) + \sum_{\tau=0}^{t-2} A^{t-2-\tau}Bu(\tau)$$

we get current state least-squares estimator:

$$\hat{x}(t-1|t-1) = A^{t-1}\hat{x}_{\mathrm{ls}}(0|t-1) + \sum_{\tau=0}^{t-2} A^{t-2-\tau}Bu(\tau)$$

righthand term ($i.e.$, effect of input on current state) is known

estimation error $z = \hat{x}(t-1|t-1) - x(t-1)$ can be expressed as

$$z = A^{t-1}\mathcal{O}_t^{\dagger} \begin{bmatrix} v(0) \\ \vdots \\ v(t-1) \end{bmatrix}$$

hence $z \ \sim \ \mathcal{N}\left(0, \sigma A^{t-1}\mathcal{O}^{\dagger}\mathcal{O}^{\dagger T}(A^{\mathsf{T}})^{t-1}\right)$

Current state estimation

$i.e.$, covariance of least-squares current state estimation error is

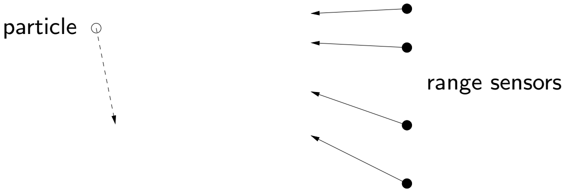
$$\sigma A^{t-1}\mathcal{O}^{\dagger}\mathcal{O}^{\dagger T}(A^{\mathsf{T}})^{t-1} = \sigma A^{t-1}\left(\sum_{\tau=0}^{t-1}(A^{\mathsf{T}})^{\tau}C^{\mathsf{T}}CA^{\tau}\right)^{-1}(A^{\mathsf{T}})^{t-1}$$

this matrix measures 'how observable' current state is, from past t inputs & outputs

- decreases (in matrix sense) as t increases
- hence has limit as $t \to \infty$ (gives limiting error covariance of estimating current state given all past inputs & outputs)

Example

- particle in $\mathbb{R}^2$ moves with uniform velocity
- (linear, noisy) range measurements from directions -15°, 0°, 20°, 30°, once per second
- range noises IID $\mathcal{N}(0,1)$; can assume RMS value of v is not much more than 2
- no assumptions about initial position & velocity



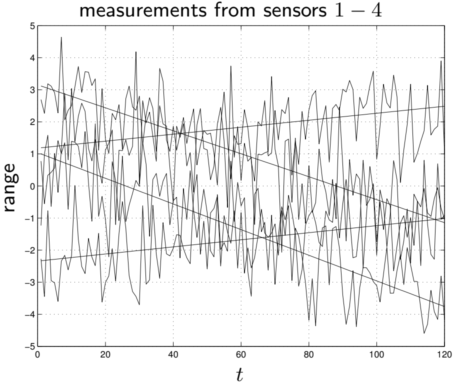
problem: estimate initial position & velocity from range measurements express as linear system

$$x(t+1) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x(t), \qquad y(t) = \begin{bmatrix} k_1^\mathsf{T} \\ \vdots \\ k_4^\mathsf{T} \end{bmatrix} x(t) + v(t)$$

- $(x_1(t), x_2(t))$ is position of particle
- $(x_3(t), x_4(t))$ is velocity of particle
- can assume RMS value of v is around 2
- k_i is unit vector from sensor i to origin

true initial position & velocities: $x(0) = (1\ \ -3\ \ -0.04\ 0.03)$

range measurements (& noiseless versions):



- estimate based on $(y(0),\ldots,y(t))$ is $\hat{x}(0|t)$
- actual RMS position error is

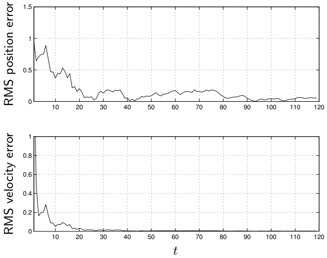
$$\sqrt{(\hat{x}_1(0|t)-x_1(0))^2+(\hat{x}_2(0|t)-x_2(0))^2}$$

(similarly for actual RMS velocity error)

- position error std. deviation is

$$\sqrt{\mathbf{E}\left((\hat{x}_1(0|t)-x_1(0))^2+(\hat{x}_2(0|t)-x_2(0))^2\right)}$$

(similarly for velocity)



Example ctd: state prediction

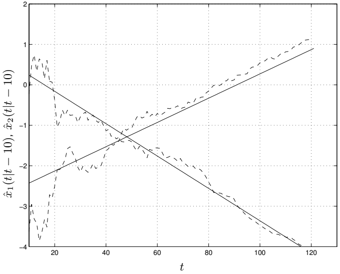
predict particle position 10 seconds in future:

how long before you can predict position 10 seconds in future with $\mathbf{E}\|\hat{x}(t+9|t-1) - x(t+9)\|^2 \leq 0.3^2$?

$$\hat{x}(t+10|t) = A^{t+10}\hat{x}_{\text{ls}}(0|t)$$

$$x(t+10) = A^{t+10}x(0)$$

plot shows estimates (dashed), and actual value (solid) of position of particle 10 steps ahead, for $10 \leq t \leq 110$



Continuous-time observability

Continuous-time observability

continuous-time system with no sensor or state noise:

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

can we deduce state x from u and y?

let's look at derivatives of y:

$$\begin{aligned} y &= Cx + Du \\ \dot{y} &= C\dot{x} + D\dot{u} = CAx + CBu + D\dot{u} \\ \ddot{y} &= CA^2x + CABu + CB\dot{u} + D\ddot{u} \end{aligned}$$

and so on

Continuous-time observability

hence we have

$$\begin{bmatrix} y \\ \dot{y} \\ \vdots \\ y^{(n-1)} \end{bmatrix} = \mathcal{O}x + \mathcal{T} \begin{bmatrix} u \\ \dot{u} \\ \vdots \\ u^{(n-1)} \end{bmatrix}$$

where $\mathcal{O}$ is the observability matrix and

$$\mathcal{T} = \begin{bmatrix} D & 0 & \cdots & & \\ CB & D & 0 & \cdots & \\ \vdots & & & & \\ CA^{n-2}B & CA^{n-3}B & \cdots & CB & D \end{bmatrix}$$

(same matrices we encountered in discrete-time case!)

rewrite as

$$\mathcal{O}x = \begin{bmatrix} y \\ \dot{y} \\ \vdots \\ y^{(n-1)} \end{bmatrix} - \mathcal{T} \begin{bmatrix} u \\ \dot{u} \\ \vdots \\ u^{(n-1)} \end{bmatrix}$$

RHS is known; x is to be determined

hence if **null**$(\mathcal{O}) = \{0\}$ we can deduce $x(t)$ from derivatives of $u(t)$, $y(t)$ up to order $n-1$

in this case we say system is observable can construct an observer using any left inverse F of $\mathcal{O}$:

$$x = F\left(\begin{bmatrix} y \\ \dot{y} \\ \vdots \\ y^{(n-1)} \end{bmatrix} - \mathcal{T}\begin{bmatrix} u \\ \dot{u} \\ \vdots \\ u^{(n-1)} \end{bmatrix}\right)$$

- reconstructs $x(t)$ (exactly and instantaneously) from

$$u(t), \ldots, u^{(n-1)}(t),\ \ y(t), \ldots, y^{(n-1)}(t)$$

- derivative-based state reconstruction is dual of state transfer using impulsive inputs

A converse

suppose $z \in \textbf{null}(\mathcal{O})$ (the unobservable subspace), and u is any input, with x, y the corresponding state and output, $i.e.$,

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

then state trajectory $\tilde{x} = x + e^{tA}z$ satisfies

$$\dot{\tilde{x}} = A\tilde{x} + Bu, \quad y = C\tilde{x} + Du$$

$i.e.$, input/output signals u, y consistent with both state trajectories x, $\tilde{x}$

hence if system is unobservable, no signal processing of any kind applied to u and y can deduce x

unobservable subspace $\textbf{null}(\mathcal{O})$ gives fundamental ambiguity in deducing x from u, y

Continuous-time least-squares state estimation

assume $\dot{x} = Ax + Bu$, $y = Cx + Du + v$ is observable

least-squares estimate of initial state $x(0)$, given $u(\tau)$, $y(\tau)$, $0 \leq \tau \leq t$: choose $\hat{x}_{\mathrm{ls}}(0)$ to minimize integral square residual

$$J = \int_0^t \left\| \tilde{y}(\tau) - Ce^{\tau A} x(0) \right\|^2 \, d\tau$$

where $\tilde{y} = y - h * u$ is observed output minus part due to input

let's expand as $J = x(0)^T Q x(0) + 2r^T x(0) + s$,

$$Q = \int_0^t e^{\tau A^T} C^T C e^{\tau A} \, d\tau, \quad r = \int_0^t e^{\tau A^T} C^T \tilde{y}(\tau) \, d\tau,$$

$$s = \int_0^t \tilde{y}(\tau)^T \tilde{y}(\tau) \, d\tau$$

setting $\nabla_{x(0)} J$ to zero, we obtain the least-squares observer

$$\hat{x}_{\mathrm{ls}}(0) = Q^{-1} r = \left(\int_0^t e^{\tau A^\mathsf{T}} C^\mathsf{T} C e^{\tau A} \, d\tau \right)^{-1} \int_0^t e^{A^\mathsf{T} \tau} C^\mathsf{T} \tilde{y}(\tau) \, d\tau$$

estimation error is

$$\tilde{x}(0) = \hat{x}_{\mathrm{ls}}(0) - x(0) = \left(\int_0^t e^{\tau A^\mathsf{T}} C^\mathsf{T} C e^{\tau A} \, d\tau \right)^{-1} \int_0^t e^{\tau A^\mathsf{T}} C^\mathsf{T} v(\tau) \, d\tau$$

therefore if $v = 0$ then $\hat{x}_{\mathrm{ls}}(0) = x(0)$

if $\frac{1}{t}\int_0^t v(\tau)^2 d\tau \leq \alpha^2$ (rms bound noise model)

set of possible estimation errors is ellipsoid $\tilde{x}(0) \in \mathcal{E}_{\mathrm{unc}}$

$$\mathcal{E}_{\mathrm{unc}} = \left\{ \left(\int_0^t e^{A^\mathsf{T}\tau} C^\mathsf{T} C e^{A\tau}\, d\tau \right)^{-1} \int_0^t e^{A^\mathsf{T}\bar{t}} C^\mathsf{T} v(\bar{t})\, d\bar{t} \;\middle|\; \frac{1}{t}\int_0^t v(\tau)^2 d\tau \leq \alpha^2 \right\}$$

maximum norm of error is

$$\|\hat{x}_{\mathrm{ls}}(0) - x(0)\| \leq \alpha \sqrt{\frac{t}{\lambda_{\mathsf{min}}(W_o(t))}}$$

where

$$W_o(t) = \int_0^t e^{A^\mathsf{T}\tau} C^\mathsf{T} C e^{A\tau}\, d\tau$$

Observability – controllability duality

let $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ be dual of system (A, B, C, D), $i.e.$,

$$\tilde{A} = A^T, \quad \tilde{B} = C^T, \quad \tilde{C} = B^T, \quad \tilde{D} = D^T$$

controllability matrix of dual system is

$$\begin{aligned} \tilde{\mathcal{C}} &= [\tilde{B} \ \tilde{A}\tilde{B} \cdots \tilde{A}^{n-1}\tilde{B}] \\ &= [C^T \ A^T C^T \cdots (A^T)^{n-1} C^T] \\ &= \mathcal{O}^T, \end{aligned}$$

transpose of observability matrix

similarly we have $\tilde{\mathcal{O}} = \mathcal{C}^T$

thus, system is observable (controllable) if and only if dual system is controllable (observable)

in fact,

$$\mathbf{null}(\mathcal{O}) = \mathbf{range}(\mathcal{O}^T)^\perp = \mathbf{range}(\tilde{\mathcal{C}})^\perp$$

$i.e.$, unobservable subspace is orthogonal complement of controllable subspace of dual