Invariant subspaces

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Invariant subspaces

Invariant subspaces

suppose $A \in \mathbb{R}^{n \times n}$ and $\mathcal{V} \subseteq \mathbb{R}^n$ is a subspace

we say that $\mathcal{V}$ is *A-invariant* if $A\mathcal{V} \subseteq \mathcal{V}$, $i.e.$, $v \in \mathcal{V} \implies Av \in \mathcal{V}$

examples:

- $\{0\}$ and $\mathbb{R}^n$ are always A-invariant
- **span**$\{v_1, \dots, v_m\}$ is A-invariant, where v_i are (right) eigenvectors of A
- if A is block upper triangular,
$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},$$
with $A_{11} \in \mathbb{R}^{r \times r}$, then $\mathcal{V} = \left\{ \begin{bmatrix} z \\ 0 \end{bmatrix} \;\middle|\; z \in \mathbb{R}^r \right\}$ is A-invariant

Examples from linear systems

- if $B \in \mathbb{R}^{n \times m}$, then the controllable subspace

  $$\textbf{range}(\mathcal{C}) = \textbf{range}\left([B \;\; AB \;\; \cdots \;\; A^{n-1}B]\right)$$

  is A-invariant

- if $C \in \mathbb{R}^{p \times n}$, then the unobservable subspace

  $$\textbf{null}(\mathcal{O}) = \textbf{null}\left(\begin{bmatrix} C \\ \vdots \\ CA^{n-1} \end{bmatrix}\right)$$

  is A-invariant

Dynamical interpretation

consider system $\dot{x} = Ax$

$\mathcal{V}$ is A-invariant if and only if

$$x(0) \in \mathcal{V} \implies x(t) \in \mathcal{V} \text{ for all } t \geq 0$$

(same statement holds for discrete-time system)

Matrix criterion for invariance

A matrix criterion for A-invariance

suppose $\mathcal{V}$ is A-invariant

let columns of $M \in \mathbb{R}^{n\times k}$ span $\mathcal{V}$, *i.e.*,

$$\mathcal{V} = \textbf{range}(M) = \textbf{range}([t_1 \ \cdots \ t_k])$$

since $At_1 \in \mathcal{V}$, we can express it as

$$At_1 = x_{11}t_1 + \cdots + x_{k1}t_k$$

we can do the same for $At_2, \ldots, At_k$, which gives

$$A[t_1 \ \cdots \ t_k] = [t_1 \ \cdots \ t_k] \begin{bmatrix} x_{11} & \cdots & x_{1k} \\ \vdots & & \vdots \\ x_{k1} & \cdots & x_{kk} \end{bmatrix}$$

or, simply, $AM = MX$

in other words: if **range**(M) is A-invariant, then there is a matrix X such that $AM = MX$

converse is also true: if there is an X such that $AM = MX$, then **range**(M) is A-invariant

now assume M is rank k, $i.e.$, $\{t_1, \ldots, t_k\}$ is a basis for $\mathcal{V}$

then every eigenvalue of X is an eigenvalue of A, and the associated eigenvector is in $\mathcal{V} =$ **range**(M)

if $Xu = \lambda u$, $u \neq 0$, then $Mu \neq 0$ and $A(Mu) = MXu = \lambda Mu$

so the eigenvalues of X are a subset of the eigenvalues of A

more generally: if $AM = MX$ (no assumption on rank of M), then A and X share at least **rank**(M) eigenvalues

Change of coordinates

Change of coordinates

suppose $\mathcal{V} = \textbf{range}(M)$ is A-invariant, where $M \in \mathbb{R}^{n \times k}$ is rank k

find $\tilde{M} \in \mathbb{R}^{n \times (n-k)}$ so that $[M\ \tilde{M}]$ is nonsingular

$$A[M\ \tilde{M}] = [AM\ A\tilde{M}] = [M\ \tilde{M}] \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}$$

where

$$\begin{bmatrix} Y \\ Z \end{bmatrix} = [M\ \tilde{M}]^{-1} A\tilde{M}$$

with $T = [M\ \tilde{M}]$, we have

$$T^{-1}AT = \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}$$

in other words: if $\mathcal{V}$ is A-invariant we can change coordinates so that

- A becomes block upper triangular in the new coordinates
- $\mathcal{V}$ corresponds to $\left\{ \begin{bmatrix} z \\ 0 \end{bmatrix} \;\middle|\; z \in \mathbb{R}^k \right\}$ in the new coordinates

Revealing controllable and unobservable subspaces

Revealing the controllable subspace

consider $\dot{x} = Ax + Bu$ (or $x_{t+1} = Ax_t + Bu_t$) and assume it is *not* controllable, so $\mathcal{V} = \textbf{range}(\mathcal{C}) \neq \mathbb{R}^n$

let columns of $M \in \mathbb{R}^k$ be basis for controllable subspace
($e.g.$, choose k independent columns from $\mathcal{C}$)

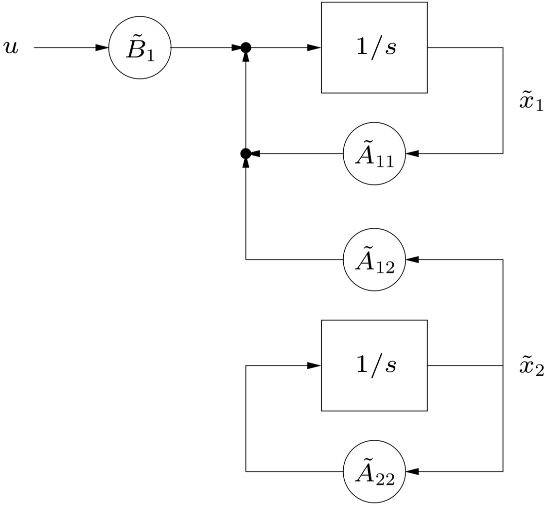
let $\tilde{M} \in \mathbb{R}^{n\times(n-k)}$ be such that $T = [M \;\; \tilde{M}]$ is nonsingular

then

$$T^{-1}AT = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix}, \qquad T^{-1}B = \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix}$$

$$\tilde{\mathcal{C}} = T^{-1}\mathcal{C} = \begin{bmatrix} \tilde{B}_1 & \cdots & \tilde{A}_{11}^{n-1}\tilde{B}_1 \\ 0 & \cdots & 0 \end{bmatrix}$$

in the new coordinates the controllable subspace is $\{(z, 0) \mid z \in \mathbb{R}^k\}$; $(\tilde{A}_{11}, \tilde{B}_1)$ is controllable we have changed coordinates to reveal the controllable subspace:



roughly speaking, $\tilde{x}_1$ is the controllable part of the state

Revealing the unobservable subspace

similarly, if (C, A) is not observable, we can change coordinates to obtain

$$T^{-1}AT = \begin{bmatrix} \tilde{A}_{11} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}, \qquad CT = \begin{bmatrix} \tilde{C}_1 & 0 \end{bmatrix}$$

and $(\tilde{C}_1, \tilde{A}_{11})$ is observable

PBH tests

Popov-Belevitch-Hautus controllability test

PBH controllability criterion: (A, B) is controllable if and only if

$$\textbf{rank}\ [sI - A\ \ B] = n \text{ for all } s \in \mathbb{C}$$

equivalent to:

(A, B) is uncontrollable if and only if there is a $w \neq 0$ with

$$w^\mathsf{T} A = \lambda w^\mathsf{T}, \qquad w^\mathsf{T} B = 0$$

$i.e.$, a left eigenvector is orthogonal to columns of B

to show it, first assume that $w \neq 0$, $w^\mathsf{T} A = \lambda w^\mathsf{T}$, $w^\mathsf{T} B = 0$

then for $k = 1, \ldots, n-1$, $w^\mathsf{T} A^k B = \lambda^k w^\mathsf{T} B = 0$, so

$$w^\mathsf{T}[B \ AB \ \cdots \ A^{n-1}B] = w^\mathsf{T}\mathcal{C} = 0$$

which shows (A, B) not controllable

conversely, suppose (A, B) not controllable

change coordinates as on p.16, let z be any left eigenvector of $\tilde{A}_{22}$, and define $\tilde{w} = (0, z)$

then $\tilde{w}^\mathsf{T}\tilde{A} = \lambda\tilde{w}^\mathsf{T}$, $\tilde{w}^\mathsf{T}\tilde{B} = 0$

it follows that $w^\mathsf{T} A = \lambda w^\mathsf{T}$, $w^\mathsf{T} B = 0$, where $w = T^{-T}\tilde{w}$

PBH observability test

PBH observability criterion: (C, A) is observable if and only if

$$\textbf{rank} \begin{bmatrix} sI - A \\ C \end{bmatrix} = n \text{ for all } s \in \mathbb{C}$$

equivalent to:

(C, A) is unobservable if and only if there is a $v \neq 0$ with

$$Av = \lambda v, \qquad Cv = 0$$

$i.e.$, a (right) eigenvector is in the nullspace of C

Observability and controllability of modes

the PBH tests allow us to identify unobservable and uncontrollable modes

the mode associated with right and left eigenvectors $v,\ w$ is

- uncontrollable if $w^\mathsf{T} B = 0$
- unobservable if $Cv = 0$

(classification can be done with repeated eigenvalues, Jordan blocks, but gets tricky)