Kalman canonical form

- lattice of invariant subspaces
- Kalman decomposition
- transfer function implications
- intrinsic eigenvalues
- hidden modes and minimal realization

Setup

consider system

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

(or $x(t+1) = Ax(t) + Bu(t)$, $y(t) = Cx(t) + Du(t)$)

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$

recall:

- *controllable subspace*: $\mathcal{R} = \textbf{range}(\mathcal{C})$, $\mathcal{C} = \left[\begin{array}{cccc} B & AB & \cdots & A^{n-1}B \end{array} \right]$
- *unobservable subspace*: $\mathcal{N} = \textbf{null}(\mathcal{O})$, $\mathcal{O} = \left[\begin{array}{c} C \\ CA \\ \vdots \\ CA^{n-1} \end{array} \right]$

question: what can we say about states that are controllable but not observable, or vice versa?

Recall: invariance and decomposition

from the invariant subspaces lecture:

- $\mathcal{R}$ and $\mathcal{N}$ are both A-invariant
- if $\mathcal{V}$ is an A-invariant subspace of dimension k, choosing a basis $T = \begin{bmatrix} T_1 & T_2 \end{bmatrix}$ with columns of T_1 spanning $\mathcal{V}$ gives
$$T^{-1}AT = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$
- this was used to reveal the controllable and unobservable subspaces separately

now: decompose the state space using *both* $\mathcal{R}$ and $\mathcal{N}$ simultaneously

Lattice of invariant subspaces

The lattice of invariant subspaces

both $\mathcal{R}$ and $\mathcal{N}$ are A-invariant

intersections and sums of A-invariant subspaces are A-invariant, so the following are all A-invariant:

$$\{0\} \subseteq \mathcal{R} \cap \mathcal{N} \subseteq \mathcal{R} \subseteq \mathcal{R} + \mathcal{N} \subseteq \mathbb{R}^n$$

$$\{0\} \subseteq \mathcal{R} \cap \mathcal{N} \subseteq \mathcal{N} \subseteq \mathcal{R} + \mathcal{N} \subseteq \mathbb{R}^n$$

this gives a lattice of five A-invariant subspaces (including $\{0\}$ and $\mathbb{R}^n$):

$$\mathbb{R}^n \supseteq \mathcal{R} + \mathcal{N}$$

$$\mathcal{R} + \mathcal{N} \supseteq \mathcal{R} \text{ and } \mathcal{R} + \mathcal{N} \supseteq \mathcal{N}$$

$$\mathcal{R} \supseteq \mathcal{R} \cap \mathcal{N} \text{ and } \mathcal{N} \supseteq \mathcal{R} \cap \mathcal{N}$$

$$\mathcal{R} \cap \mathcal{N} \supseteq \{0\}$$

Choosing a basis adapted to the lattice

to construct the Kalman decomposition, choose *any* basis adapted to the chain of subspaces:

step 1: choose a basis $T_{c\bar{o}}$ for $\mathcal{R} \cap \mathcal{N}$

step 2: extend to a basis for $\mathcal{R}$ by adding vectors T_{co}

step 3: extend to a basis for $\mathcal{N}$ by adding vectors $T_{\bar{c}\bar{o}}$

step 4: extend to a basis for $\mathcal{R} + \mathcal{N}$ by adding vectors $T_{\bar{c}o}$

step 5: if $\mathcal{R} + \mathcal{N} \neq \mathbb{R}^n$, extend to a basis for $\mathbb{R}^n$ (append these to $T_{\bar{c}o}$)

set $T = \begin{bmatrix} T_{co} & T_{c\bar{o}} & T_{\bar{c}o} & T_{\bar{c}\bar{o}} \end{bmatrix}$

A-invariant complements are not needed

important: the four sets of basis vectors T_{co}, $T_{c\bar{o}}$, $T_{\bar{c}o}$, $T_{\bar{c}\bar{o}}$ are *not* required to span A-invariant subspaces

only the subspaces in the lattice ($\mathcal{R} \cap \mathcal{N}$, $\mathcal{R}$, $\mathcal{N}$, $\mathcal{R} + \mathcal{N}$) need to be A-invariant — and they always are

the block structure of $\bar{A}$ comes entirely from the A-invariance of these subspaces, not from any property of the complements

in fact, A-invariant complements may not exist: $e.g.$, for $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, the subspace **span** $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ is A-invariant but has no A-invariant complement (A has only one eigenvector)

Kalman decomposition

Kalman decomposition

with $T = \begin{bmatrix} T_{co} & T_{c\bar{o}} & T_{\bar{c}o} & T_{\bar{c}\bar{o}} \end{bmatrix}$ chosen as above, let $\bar{x} = T^{-1}x$, partitioned as $\bar{x} = \begin{bmatrix} \bar{x}_{co} \\ \bar{x}_{c\bar{o}} \\ \bar{x}_{\bar{c}o} \\ \bar{x}_{\bar{c}\bar{o}} \end{bmatrix}$

the transformed system has the structure:

$$\bar{A} = T^{-1}AT = \begin{bmatrix} A_{co} & 0 & A_{13} & 0 \\ A_{21} & A_{c\bar{o}} & A_{23} & A_{24} \\ 0 & 0 & A_{\bar{c}o} & 0 \\ 0 & 0 & A_{43} & A_{\bar{c}\bar{o}} \end{bmatrix}$$

$$\bar{B} = T^{-1}B = \begin{bmatrix} B_{co} \\ B_{c\bar{o}} \\ 0 \\ 0 \end{bmatrix}, \quad \bar{C} = CT = \begin{bmatrix} C_{co} & 0 & C_{\bar{c}o} & 0 \end{bmatrix}$$

Why the zero blocks appear

every zero block follows from the A-invariance of a subspace in the lattice, or from **range**$(B) \subseteq \mathcal{R}$, or $\mathcal{N} \subseteq$ **null**(C):

- rows 3–4, columns 1–2 of $\bar{A}$ are zero:

  A maps **range**$(T_{co}, T_{c\bar{o}}) = \mathcal{R}$ into $\mathcal{R}$, so the image has no component in the $T_{\bar{c}o}$, $T_{\bar{c}\bar{o}}$ directions

- rows 1 and 3, columns 2 and 4 of $\bar{A}$ are zero:

  A maps **range**$(T_{c\bar{o}}, T_{\bar{c}\bar{o}}) = \mathcal{N}$ into $\mathcal{N}$, so the image has no component in the T_{co}, $T_{\bar{c}o}$ directions

- $\bar{B}$ has zeros in rows 3–4: **range**$(B) \subseteq \mathcal{R}$

- $\bar{C}$ has zeros in columns 2 and 4: $\mathcal{N} \subseteq$ **null**(C)

the complement vectors T_{co}, $T_{\bar{c}o}$ can be arbitrary — the zero pattern is unchanged

Kalman decomposition: subsystems

$$
\begin{aligned}
\dot{\bar{x}}_{co} &= A_{co}\,\bar{x}_{co} + A_{13}\,\bar{x}_{\bar{c}o} + B_{co}\,u \\
\dot{\bar{x}}_{c\bar{o}} &= A_{21}\,\bar{x}_{co} + A_{c\bar{o}}\,\bar{x}_{c\bar{o}} + A_{23}\,\bar{x}_{\bar{c}o} + A_{24}\,\bar{x}_{\bar{c}\bar{o}} + B_{c\bar{o}}\,u \\
\dot{\bar{x}}_{\bar{c}o} &= A_{\bar{c}o}\,\bar{x}_{\bar{c}o} \\
\dot{\bar{x}}_{\bar{c}\bar{o}} &= A_{43}\,\bar{x}_{\bar{c}o} + A_{\bar{c}\bar{o}}\,\bar{x}_{\bar{c}\bar{o}}
\end{aligned}
$$

$$
y = C_{co}\,\bar{x}_{co} + C_{\bar{c}o}\,\bar{x}_{\bar{c}o} + Du
$$

- uncontrollable part ($\bar{x}_{\bar{c}o}$, $\bar{x}_{\bar{c}\bar{o}}$) evolves autonomously (no input)
- output depends only on observable part ($\bar{x}_{co}$, $\bar{x}_{\bar{c}o}$)

Transfer function

Transfer function

the transfer function is

$$G(s) = C(sI - A)^{-1}B + D$$

this is invariant under change of coordinates:

$$\begin{aligned}
\bar{C}(sI - \bar{A})^{-1}\bar{B} + D &= CT\left(sI - T^{-1}AT\right)^{-1}T^{-1}B + D \\
&= CT\left(T^{-1}(sI - A)T\right)^{-1}T^{-1}B + D \\
&= CTT^{-1}(sI - A)^{-1}TT^{-1}B + D \\
&= C(sI - A)^{-1}B + D
\end{aligned}$$

hence transfer function is the same in Kalman coordinates

Transfer function in Kalman form

using block structure of $\bar{A}$, $\bar{B}$, $\bar{C}$:

input u only enters controllable states ($\bar{x}_{co}$, $\bar{x}_{c\bar{o}}$)

output y only depends on observable states ($\bar{x}_{co}$, $\bar{x}_{\bar{c}o}$)

hence only $\bar{x}_{co}$ (controllable *and* observable) contributes to $G(s)$:

$$G(s) = C_{co}(sI - A_{co})^{-1} B_{co} + D$$

proof: with $\bar{x}(0) = 0$, uncontrollable states remain zero, so $\bar{x}_{\bar{c}o}(t) = 0$ and $\bar{x}_{\bar{c}\bar{o}}(t) = 0$

output is then $y = C_{co}\,\bar{x}_{co} + Du$ with $\dot{\bar{x}}_{co} = A_{co}\,\bar{x}_{co} + B_{co}\,u$

Intrinsic eigenvalues

Eigenvalues of restricted maps

recall from invariant subspaces: if $\mathcal{V}$ is A-invariant and M is a basis for $\mathcal{V}$, then $AM = MX$

the eigenvalues of X depend only on $\mathcal{V}$ (not on M): a different basis $\tilde{M} = MS$ gives $A\tilde{M} = \tilde{M}(S^{-1}XS)$, similar to X

we call these "the eigenvalues of A restricted to $\mathcal{V}$"

every eigenvalue of X is an eigenvalue of A (if $Xu = \lambda u$, $u \neq 0$, then $A(Mu) = \lambda(Mu)$)

Intrinsic eigenvalues of the Kalman blocks

the restriction of A to $\mathcal{R}$ is represented in the Kalman basis by $\begin{bmatrix} A_{co} & 0 \\ A_{21} & A_{c\bar{o}} \end{bmatrix}$ (block triangular)

its eigenvalues are those of A_{co} together with those of $A_{c\bar{o}}$

but these eigenvalues are *intrinsic*: they depend only on $\mathcal{R}$ and A, not on the complement vectors T_{co}

similarly, the restriction of A to $\mathcal{R} \cap \mathcal{N}$ gives the eigenvalues of $A_{c\bar{o}}$ (intrinsic, since $\mathcal{R} \cap \mathcal{N}$ is A-invariant)

hence: eigenvalues of A_{co} = eigenvalues of $A|_{\mathcal{R}}$ minus eigenvalues of $A|_{\mathcal{R} \cap \mathcal{N}}$

the complement T_{co} changes the matrix A_{co} and the coupling A_{21}, but not the eigenvalues of A_{co}

All four blocks are intrinsic

the same argument applied to each branch of the lattice:

- eigenvalues of $A_{c\bar{o}}$: eigenvalues of $A|_{\mathcal{R}\cap\mathcal{N}}$ (directly intrinsic)
- eigenvalues of A_{co}: eigenvalues of $A|_{\mathcal{R}}$ minus those of $A|_{\mathcal{R}\cap\mathcal{N}}$
- eigenvalues of $A_{\bar{c}\bar{o}}$: eigenvalues of $A|_{\mathcal{N}}$ minus those of $A|_{\mathcal{R}\cap\mathcal{N}}$
- eigenvalues of $A_{\bar{c}o}$: eigenvalues of $A|_{\mathcal{R}+\mathcal{N}}$ minus those of $A|_{\mathcal{R}}$ minus those of $A|_{\mathcal{N}}$ plus those of $A|_{\mathcal{R}\cap\mathcal{N}}$ (inclusion-exclusion on the lattice)

each step uses only the A-invariance of subspaces in the lattice, the block triangular eigenvalue fact, and the basis-independence of eigenvalues of restricted maps

the poles of $G(s)$ (eigenvalues of A_{co}) and the hidden modes are intrinsic properties of the system

Implications

- the transfer function $G(s)$ depends only on the controllable and observable subsystem $(A_{co}, B_{co}, C_{co}, D)$
- eigenvalues of A_{co} are the poles of $G(s)$; remaining eigenvalues of A are *hidden modes*
- if (A, B) is controllable and (A, C) is observable, then $A_{co} = A$ and $G(s) = C(sI - A)^{-1}B + D$ has n poles

Hidden modes and minimal realization

Hidden modes

example: $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $D = 0$

transfer function:

$$G(s) = C(sI - A)^{-1}B = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{s-1} & 0 \\ 0 & \frac{1}{s-2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{s-1}$$

controllability matrix: $\mathcal{C} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, $\textbf{rank}(\mathcal{C}) = 2$ (controllable)

observability matrix: $\mathcal{O} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $\textbf{rank}(\mathcal{O}) = 1$ (not observable)

unobservable subspace is $\mathcal{N} = \textbf{span}\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

eigenvalue $s = 2$ is a hidden mode: does not appear in $G(s)$

Hidden modes ctd

in Kalman coordinates with $T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (already diagonal):

$\bar{x}_{co} = x_1$ (controllable and observable), $\bar{x}_{c\bar{o}} = x_2$ (controllable, unobservable)

$$\begin{aligned} \dot{x}_1 &= x_1 + u \\ \dot{x}_2 &= 2x_2 + u \\ y &= x_1 \end{aligned}$$

transfer function: $G(s) = \frac{1}{s-1}$

the mode at $s = 2$ is excited by u but is invisible in y

Minimal realization

a state-space model (A, B, C, D) is called a *minimal realization* of $G(s)$ if no lower-dimensional state-space model gives the same $G(s)$

fact: (A, B, C, D) is a minimal realization of $G(s)$ if and only if (A, B) is controllable and (A, C) is observable

proof ($\Rightarrow$): if (A, B) is not controllable or (A, C) is not observable, then $G(s) = C_{co}(sI - A_{co})^{-1}B_{co} + D$ with $\mathbf{dim}(A_{co}) < n$

hence realization is not minimal

proof ($\Leftarrow$): if (A, B) is controllable and (A, C) is observable, then $A_{co} = A$ (dimension n)

any realization of $G(s)$ has dimension $\geq n$ since $G(s)$ has n poles, so the realization is minimal

Summary

- Kalman decomposition separates state space into four parts: controllable–observable, controllable–unobservable, uncontrollable–observable, uncontrollable–unobservable
- relies on A-invariance of the lattice $\mathcal{R} \cap \mathcal{N} \subseteq \mathcal{R}, \mathcal{N} \subseteq \mathcal{R} + \mathcal{N}$
- complement vectors can be arbitrary; A-invariant complements are not needed (and may not exist)
- the zero pattern in $\bar{A}$, $\bar{B}$, $\bar{C}$ comes entirely from the invariant subspaces
- eigenvalues of each diagonal block are intrinsic (determined by quotient maps, independent of complement choice)
- transfer function determined solely by controllable and observable subsystem
- minimal realization $\Leftrightarrow$ controllable and observable