Pole placement by state feedback

- state feedback and eigenvalue assignment
- controllable canonical form
- Ackermann's formula (single input)
- extension to multiple inputs

State feedback

consider system

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

(or $x(t+1) = Ax(t) + Bu(t)$, $y(t) = Cx(t) + Du(t)$)

state feedback is $u = Kx$ with $K \in \mathbb{R}^{m \times n}$

closed-loop system:

$$\dot{x} = (A + BK)x$$

eigenvalues of $A + BK$ determine closed-loop behavior ($e.g.$, stability, transient response)

pole placement problem: given desired eigenvalues $\lambda_1, \ldots, \lambda_n$, find K such that

$$\mathbf{det}(sI - A - BK) = (s - \lambda_1)(s - \lambda_2) \cdots (s - \lambda_n)$$

Pole placement and controllability

fact: arbitrary pole placement by state feedback is possible if and only if (A, B) is controllable

$i.e.$, for any choice of $\lambda_1, \ldots, \lambda_n$ (closed under complex conjugation), there exists K such that eigenvalues of $A + BK$ are $\lambda_1, \ldots, \lambda_n$

- we prove this constructively for the single-input case via Ackermann's formula
- for $m = 1$: $K \in \mathbb{R}^{1\times n}$ is unique
- for $m > 1$: K is not unique

Controllable canonical form

Suppose (A, B) is a controllable pair, and B is a column vector. Then there exists a state transformation T such that

$$T^{-1}AT = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \\ -a_0 & -a_1 & & -a_{n-1} \end{bmatrix} \quad \text{and} \quad T^{-1}B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

- characteristic polynomial of $T^{-1}AT$ (and therefore also A) is

$$s^n + a_{n-1}s^{n-1} + \cdots + a_0$$

- $T^{-1}AT$ is called a *companion matrix*

Constructing the transformation

- Cayley-Hamilton gives

$$A^n + a_{n-1}A^{n-1} + \cdots + a_0 I = 0 \ .$$

- multiplying by B

$$A^n B = -a_0 B - \cdots - a_{n-1}A^{n-1}B \ .$$

- and so (note that P is square and nonsingular)

$$A \begin{bmatrix} B & \cdots & A^{n-1}B \end{bmatrix} = \underbrace{\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 0 & & 0 & -a_0 \\ 1 & \ddots & & -a_1 \\ & \ddots & 0 & \vdots \\ 0 & & 1 & -a_{n-1} \end{bmatrix}}_{M}$$

- which gives $P^{-1}AP = M$

Constructing the transformation

- now let

$$\tilde{A} = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \\ -a_0 & -a_1 & & -a_{n-1} \end{bmatrix} \qquad \text{and} \qquad \tilde{B} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

- define $\tilde{P} := \begin{bmatrix} \tilde{B} & \tilde{A}\tilde{B} & \cdots & \tilde{A}^{n-1}\tilde{B} \end{bmatrix}$
- one can check $(\tilde{A}, \tilde{B})$ is controllable, so the approach from the previous slide gives

$$\tilde{P}^{-1}\tilde{A}\tilde{P} = M \qquad \text{and} \qquad \tilde{P}^{-1}\tilde{B} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

- therefore for the original A, B we have

$$P^{-1}AP = \tilde{P}^{-1}\tilde{A}\tilde{P} \qquad \text{and} \qquad P^{-1}B = \tilde{P}^{-1}\tilde{B}$$

- set $T = P\tilde{P}^{-1}$

Pole placement in canonical form

- suppose A, B are in controllable canonical form, and $B \in \mathbb{R}^{n\times 1}$
- use feedback $u = Kx$
- $K = \begin{bmatrix} k_0 & k_1 & \dots & k_{n-1} \end{bmatrix}$
- closed-loop matrix $A + BK$ differs from A only in the last row

$$A + BK = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 + k_0 & -a_1 + k_1 & \cdots & \cdots & -a_{n-1} + k_{n-1} \end{bmatrix}$$

the characteristic polynomial is

$$\mathbf{det}(sI - A - BK) = s^n + (a_{n-1} - k_{n-1})s^{n-1} + \cdots + (a_0 - k_0)$$

so by choosing K we can set this to be equal to $(s - \lambda_1)(s - \lambda_2)\cdots(s - \lambda_n)$

Transforming back

$$K = -e_n^\mathsf{T} \mathcal{C}^{-1} \alpha(A)$$

this is *Ackermann's formula*

Ackermann's formula

theorem (Ackermann): given controllable (A, B) with $B \in \mathbb{R}^{n\times 1}$, the unique state feedback $u = Kx$ such that

$$\mathbf{det}(sI - A - BK) = \alpha(s)$$

is given by

$$K = -e_n^\mathsf{T} \mathcal{C}^{-1} \alpha(A)$$

where

- $\mathcal{C} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$ is the controllability matrix
- $e_n^\mathsf{T} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$
- $\alpha(A) = A^n + \alpha_{n-1}A^{n-1} + \cdots + \alpha_0 I$

Row structure of T^{-1}

let R_j denote row j of T^{-1}

claim: $R_j = R_1 A^{j-1}$, $i.e.$ all rows are determined by the first

proof: the transformation satisfies $T^{-1}AT = \tilde{A}$, equivalently

$$T^{-1}A = \tilde{A}T^{-1}$$

take row j of both sides:

- row j of $T^{-1}A$ is $R_j A$
- for $j < n$, row j of $\tilde{A}$ is e_{j+1}^T (the superdiagonal entry), so row j of $\tilde{A}T^{-1}$ is $e_{j+1}^\mathsf{T}T^{-1} = R_{j+1}$

therefore $R_j A = R_{j+1}$ for $j = 1, \ldots, n-1$

applying this repeatedly: $R_j = R_1 A^{j-1}$ as desired

Row structure of T^{-1} ctd

claim: $R_1 = e_n^\mathsf{T} \mathcal{C}^{-1}$, and therefore $R_j = e_n^\mathsf{T} \mathcal{C}^{-1} A^{j-1}$

proof: the transformation satisfies $T^{-1}B = e_n$, so

$$R_j B = (e_n)_j = \delta_{j,n}$$

substituting $R_j = R_1 A^{j-1}$ gives $R_1 A^{j-1} B = \delta_{j,n}$ for each j

and assembling this into a matrix gives

$$R_1 \underbrace{\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}}_{\mathcal{C}} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} = e_n^\mathsf{T}$$

so $R_1 = e_n^\mathsf{T} \mathcal{C}^{-1}$, and therefore

$$R_j = R_1 A^{j-1} = e_n^\mathsf{T} \mathcal{C}^{-1} A^{j-1}$$

Proof of Ackermann's formula

- from above, to achieve $\mathbf{det}(sI - \tilde{A} - \tilde{B}\tilde{K}) = \alpha(s)$ in companion-form coordinates, we need

$$\tilde{K} = \begin{bmatrix} a_0 - \alpha_0 & a_1 - \alpha_1 & \cdots & a_{n-1} - \alpha_{n-1} \end{bmatrix}$$

- in the original coordinates the gain is $K = \tilde{K}T^{-1}$
- using $R_j = e_n^\mathsf{T} \mathcal{C}^{-1} A^{j-1}$ we have

$$K = \tilde{K}T^{-1} = \sum_{j=0}^{n-1} (a_j - \alpha_j)\, e_n^\mathsf{T} \mathcal{C}^{-1} A^j = e_n^\mathsf{T} \mathcal{C}^{-1} \underbrace{\sum_{j=0}^{n-1} (a_j - \alpha_j) A^j}_{= \chi(A) - \alpha(A)}$$

- by Cayley-Hamilton $\chi(A) = 0$, so

$$K = -e_n^\mathsf{T} \mathcal{C}^{-1} \alpha(A)$$

Example

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

open-loop eigenvalues are $\lambda = -1, -2$

desired eigenvalues are $\lambda = -3, -4$, so $\alpha(s) = (s+3)(s+4) = s^2 + 7s + 12$

controllability matrix $\mathcal{C} = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix}$, $\mathcal{C}^{-1} = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$

compute $\alpha(A)$

$$\alpha(A) = A^2 + 7A + 12I = \begin{bmatrix} -2 & -3 \\ 6 & 7 \end{bmatrix} + 7\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} + 12I = \begin{bmatrix} 10 & 4 \\ -8 & -2 \end{bmatrix}$$

Ackermann's formula gives

$$K = -e_2^T \mathcal{C}^{-1} \alpha(A) = -\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 10 & 4 \\ -8 & -2 \end{bmatrix} = \begin{bmatrix} -10 & -4 \end{bmatrix}$$

Example ctd

we can check

$$A + BK = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} -10 & -4 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -12 & -7 \end{bmatrix}$$

$$\mathbf{det}(sI - A - BK) = s^2 + 7s + 12 = (s+3)(s+4)$$

Multiple inputs

now consider $B \in \mathbb{R}^{n \times m}$ with $m > 1$

pole placement problem: find $K \in \mathbb{R}^{m \times n}$ such that $\mathbf{det}(sI - A - BK) = \alpha(s)$

- if (A, B) is controllable, a solution exists
- K is no longer unique ($m \times n$ unknowns, only n equations from characteristic polynomial)
- extra degrees of freedom can be used for other objectives ($e.g.$, robustness, gain magnitude)

approach is to reduce to single-input problem

Reduction to single input

Suppose (A, B) is controllable and $B_1 \in \mathbb{R}^{n\times 1}$ is any nonzero column of B. Then there exists an F such that the matrix pair $(A + BF, B_1)$ is controllable.

- we omit the proof
- once we have F, we can apply the single input approach
- which gives K such that $A + BF + B_1K$ has the desired eigenvalues, and

$$A + BF + B_1K = A + B\left(F + \begin{bmatrix} K \\ 0 \end{bmatrix}\right)$$

(assuming that B_1 is the first column of B)

Necessity of controllability

fact: if (A, B) is not controllable, arbitrary pole placement is not possible

proof: by Kalman decomposition, in suitable coordinates, we have

$$\bar{A} = \begin{bmatrix} A_c & A_{12} \\ 0 & A_{\bar{c}} \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B_c \\ 0 \end{bmatrix}$$

and so for any K

$$\bar{A} + \bar{B}K = \begin{bmatrix} A_c + B_c K_1 & A_{12} + B_c K_2 \\ 0 & A_{\bar{c}} \end{bmatrix}$$

eigenvalues of $A_{\bar{c}}$ are always eigenvalues of $\bar{A} + \bar{B}K$, regardless of K

hence eigenvalues of $A_{\bar{c}}$ cannot be moved by state feedback; these are the *uncontrollable modes*

Summary

- state feedback $u = Kx$ places eigenvalues of $A + BK$
- arbitrary placement possible iff (A, B) is controllable
- single input ($m = 1$): Ackermann's formula gives unique K:

$$K = -e_n^\mathsf{T} \mathcal{C}^{-1} \alpha(A)$$

- multiple inputs ($m > 1$): reduce to single-input case
- uncontrollable modes cannot be moved by any feedback