

Infinite horizon linear quadratic regulator

- ▶ infinite horizon LQR problem
- ▶ dynamic programming solution
- ▶ receding horizon LQR control
- ▶ closed-loop system

Infinite horizon LQR problem

discrete-time system $x_{t+1} = Ax_t + Bu_t$, $x_0 = x^{\text{init}}$

problem: choose u_0, u_1, \dots to minimize

$$J = \sum_{\tau=0}^{\infty} (x_{\tau}^T Q x_{\tau} + u_{\tau}^T R u_{\tau})$$

with given constant state and input weight matrices

$$Q = Q^T \geq 0, \quad R = R^T > 0$$

... an *infinite dimensional problem*

problem: it's possible that $J = \infty$ for all input sequences u_0, \dots

$$x_{t+1} = 2x_t + 0u_t, \quad x^{\text{init}} = 1$$

let's assume (A, B) is controllable

then for any x^{init} there's an input sequence

$$u_0, \dots, u_{n-1}, 0, 0, \dots$$

that steers x to zero at $t = n$, and keeps it there

for this u , $J < \infty$

and therefore, $\min_u J < \infty$ for any x^{init}

Dynamic programming solution

define **value function** $V : \mathbb{R}^n \rightarrow \mathbb{R}$

$$V(z) = \min_{u_0, \dots} \sum_{\tau=0}^{\infty} (x_{\tau}^{\top} Q x_{\tau} + u_{\tau}^{\top} R u_{\tau})$$

subject to $x_0 = z$, $x_{\tau+1} = Ax_{\tau} + Bu_{\tau}$

- ▶ $V(z)$ is the minimum LQR cost-to-go, starting from state z
- ▶ doesn't depend on time-to-go, which is always ∞ ; infinite horizon problem is *shift invariant*

Hamilton-Jacobi equation

fact: V is quadratic, *i.e.*, $V(z) = z^T P z$, where $P = P^T \geq 0$
(can be argued directly from first principles)

HJ equation:

$$V(z) = \min_w (z^T Q z + w^T R w + V(Az + Bw))$$

or

$$z^T P z = \min_w (z^T Q z + w^T R w + (Az + Bw)^T P (Az + Bw))$$

minimizing w is $w^* = -(R + B^T P B)^{-1} B^T P A z$

so HJ equation is

$$\begin{aligned} z^T P z &= z^T Q z + w^{*T} R w^* + (Az + Bw^*)^T P (Az + Bw^*) \\ &= z^T (Q + A^T P A - A^T P B (R + B^T P B)^{-1} B^T P A) z \end{aligned}$$

this must hold for all z , so we conclude that P satisfies the ARE

$$P = Q + A^T P A - A^T P B (R + B^T P B)^{-1} B^T P A$$

and the optimal input is constant state feedback $u_t = K x_t$,

$$K = -(R + B^T P B)^{-1} B^T P A$$

compared to finite-horizon LQR problem,

- ▶ value function and optimal state feedback gains are time-invariant
- ▶ we don't have a recursion to compute P ; we only have the ARE

fact: the ARE has only one positive semidefinite solution P

i.e., ARE plus $P = P^T \geq 0$ uniquely characterizes value function

consequence: the Riccati recursion

$$P_{k+1} = Q + A^T P_k A - A^T P_k B (R + B^T P_k B)^{-1} B^T P_k A, \quad P_1 = Q$$

converges to the unique PSD solution of the ARE

(when (A, B) controllable)

(later we'll see direct methods to solve ARE)

thus, infinite-horizon LQR optimal control is same as steady-state finite horizon optimal control

Receding-horizon LQR control

consider cost function

$$J_t(u_t, \dots, u_{t+T-1}) = \sum_{\tau=t}^{\tau=t+T} (x_\tau^\top Q x_\tau + u_\tau^\top R u_\tau)$$

- ▶ T is called *horizon*
- ▶ same as infinite horizon LQR cost, truncated after T steps into future

if $(u_t^*, \dots, u_{t+T-1}^*)$ minimizes J_t , u_t^* is called (T -step ahead) *optimal receding horizon control* in words:

- ▶ at time t , find input sequence that minimizes T -step-ahead LQR cost, starting at current time
- ▶ then use only the first input

example: 1-step ahead receding horizon control

find u_t, u_{t+1} that minimize

$$J_t = x_t^T Q x_t + x_{t+1}^T Q x_{t+1} + u_t^T R u_t + u_{t+1}^T R u_{t+1}$$

first term doesn't matter; optimal choice for u_{t+1} is 0; optimal u_t minimizes

$$x_{t+1}^T Q x_{t+1} + u_t^T R u_t = (Ax_t + Bu_t)^T Q (Ax_t + Bu_t) + u_t^T R u_t$$

thus, 1-step ahead receding horizon optimal input is

$$u_t = -(R + B^T Q B)^{-1} B^T Q A x_t$$

... a constant state feedback

in general, optimal T -step ahead LQR control is

$$u_t = K_T x_t, \quad K_T = -(R + B^T P_T B)^{-1} B^T P_T A$$

where

$$P_1 = Q, \quad P_{i+1} = Q + A^T P_i A - A^T P_i B (R + B^T P_i B)^{-1} B^T P_i A$$

i.e.: same as the optimal finite horizon LQR control, $T - 1$ steps before the horizon N

- ▶ a constant state feedback
- ▶ state feedback gain converges to infinite horizon optimal as horizon becomes long (assuming controllability)

Closed-loop system

suppose K is LQR-optimal state feedback gain

$$x_{t+1} = Ax_t + Bu_t = (A + BK)x_t$$

is called *closed-loop system* ($x_{t+1} = Ax_t$ is called *open-loop system*)

is closed-loop system stable? consider

$$x_{t+1} = 2x_t + u_t, \quad Q = 0, \quad R = 1$$

optimal control is $u_t = 0x_t$, *i.e.*, closed-loop system is unstable

fact: if (Q, A) observable and (A, B) controllable, then closed-loop system is stable