

Continuous time linear quadratic regulator

- ▶ continuous-time LQR problem
- ▶ dynamic programming solution
- ▶ Hamiltonian system and two point boundary value problem
- ▶ infinite horizon LQR
- ▶ direct solution of ARE via Hamiltonian

Continuous-time LQR problem

continuous-time system $\dot{x} = Ax + Bu$, $x(0) = x_0$

problem: choose $u : [0, T] \rightarrow \mathbb{R}^m$ to minimize

$$J = \int_0^T (x(\tau)^T Q x(\tau) + u(\tau)^T R u(\tau)) d\tau + x(T)^T Q_f x(T)$$

▶ T is *time horizon*

▶ $Q = Q^T \geq 0$, $Q_f = Q_f^T \geq 0$, $R = R^T > 0$ are *state cost*, *final state cost*, and *input cost* matrices

... an *infinite-dimensional problem*: (trajectory $u : [0, T] \rightarrow \mathbb{R}^m$ is variable)

Dynamic programming solution

we'll solve LQR problem using dynamic programming

for $0 \leq t \leq T$ we define the **value function** $V_t : \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$V_t(z) = \min_u \int_t^T (x(\tau)^\top Q x(\tau) + u(\tau)^\top R u(\tau)) d\tau + x(T)^\top Q_f x(T)$$

subject to $x(t) = z$, $\dot{x} = Ax + Bu$

- ▶ minimum is taken over all possible signals $u : [t, T] \rightarrow \mathbb{R}^m$
- ▶ $V_t(z)$ gives the minimum LQR cost-to-go, starting from state z at time t
- ▶ $V_T(z) = z^\top Q_f z$

fact: V_t is quadratic, *i.e.*, $V_t(z) = z^T P_t z$, where $P_t = P_t^T \geq 0$

similar to discrete-time case:

- ▶ P_t can be found from a *differential equation* running backward in time from $t = T$
- ▶ the LQR optimal u is easily expressed in terms of P_t

we start with $x(t) = z$

let's take $u(t) = w \in \mathbb{R}^m$, a constant, over the time interval $[t, t + h]$, where $h > 0$ is small

cost incurred over $[t, t + h]$ is

$$\int_t^{t+h} (x(\tau)^\top Q x(\tau) + w^\top R w) d\tau \approx h(z^\top Q z + w^\top R w)$$

and we end up at $x(t + h) \approx z + h(Az + Bw)$

min-cost-to-go from where we land is approximately

$$\begin{aligned}V_{t+h}(z + h(Az + Bw)) &= (z + h(Az + Bw))^T P_{t+h}(z + h(Az + Bw)) \\ &\approx (z + h(Az + Bw))^T (P_t + h\dot{P}_t)(z + h(Az + Bw)) \\ &\approx z^T P_t z + h \left((Az + Bw)^T P_t z + z^T P_t (Az + Bw) + z^T \dot{P}_t z \right)\end{aligned}$$

(dropping h^2 and higher terms)

cost incurred plus min-cost-to-go is approximately

$$z^T P_t z + h \left(z^T Q z + w^T R w + (Az + Bw)^T P_t z + z^T P_t (Az + Bw) + z^T \dot{P}_t z \right)$$

minimize over w to get (approximately) optimal w , which must satisfy $2hw^T R + 2hz^T P_t B = 0$

so

$$w^* = -R^{-1} B^T P_t z$$

thus optimal u is time-varying linear state feedback:

$$u_{\text{qr}}(t) = K_t x(t), \quad K_t = -R^{-1} B^T P_t$$

HJ equation

now let's substitute w^* into HJ equation:

$$z^T P_t z \approx z^T P_t z + \\ + h (z^T Q z + w^{*T} R w^* + (Az + Bw^*)^T P_t z + z^T P_t (Az + Bw^*) + z^T \dot{P}_t z)$$

yields, after simplification,

$$-\dot{P}_t = A^T P_t + P_t A - P_t B R^{-1} B^T P_t + Q$$

which is the *Riccati differential equation* for the LQR problem

we can solve it (numerically) using the *final condition* $P_T = Q_f$

Summary of cts-time LQR solution via DP

1. solve Riccati differential equation

$$-\dot{P}_t = A^\top P_t + P_t A - P_t B R^{-1} B^\top P_t + Q, \quad P_T = Q_f$$

(backward in time)

2. optimal u is $u_{\text{lqr}}(t) = K_t x(t)$, $K_t := -R^{-1} B^\top P_t$

DP method readily extends to time-varying A , B , Q , R , and tracking problem

Steady-state regulator

usually P_t rapidly converges as t decreases below T

limit P_{ss} satisfies (cts-time) algebraic Riccati equation (ARE)

$$A^T P + PA - PBR^{-1}B^T P + Q = 0$$

a quadratic matrix equation

- ▶ P_{ss} can be found by (numerically) integrating the Riccati differential equation, or by direct methods
- ▶ for t not close to horizon T , LQR optimal input is approximately a linear, constant state feedback

$$u(t) = K_{ss}x(t), \quad K_{ss} = -R^{-1}B^T P_{ss}$$

Derivation via discretization

let's discretize using small step size $h > 0$, with $Nh = T$

$$x((k+1)h) \approx x(kh) + h\dot{x}(kh) = (I + hA)x(kh) + hBu(kh)$$

$$J \approx \frac{h}{2} \sum_{k=0}^{N-1} (x(kh)^\top Q x(kh) + u(kh)^\top R u(kh)) + \frac{1}{2} x(Nh)^\top Q_f x(Nh)$$

this yields a discrete-time LQR problem, with parameters

$$\tilde{A} = I + hA, \quad \tilde{B} = hB, \quad \tilde{Q} = hQ, \quad \tilde{R} = hR, \quad \tilde{Q}_f = Q_f$$

solution to discrete-time LQR problem is $u(kh) = \tilde{K}_k x(kh)$,

$$\tilde{K}_k = -(\tilde{R} + \tilde{B}^\top \tilde{P}_{k+1} \tilde{B})^{-1} \tilde{B}^\top \tilde{P}_{k+1} \tilde{A}$$

$$\tilde{P}_{k-1} = \tilde{Q} + \tilde{A}^\top \tilde{P}_k \tilde{A} - \tilde{A}^\top \tilde{P}_k \tilde{B} (\tilde{R} + \tilde{B}^\top \tilde{P}_k \tilde{B})^{-1} \tilde{B}^\top \tilde{P}_k \tilde{A}$$

substituting and keeping only h^0 and h^1 terms yields

$$\tilde{P}_{k-1} = hQ + \tilde{P}_k + hA^\top \tilde{P}_k + h\tilde{P}_k A - h\tilde{P}_k B R^{-1} B^\top \tilde{P}_k$$

which is the same as

$$-\frac{1}{h}(\tilde{P}_k - \tilde{P}_{k-1}) = Q + A^\top \tilde{P}_k + \tilde{P}_k A - \tilde{P}_k B R^{-1} B^\top \tilde{P}_k$$

letting $h \rightarrow 0$ we see that $\tilde{P}_k \rightarrow P_{kh}$, where

$$-\dot{P} = Q + A^\top P + PA - PBR^{-1}B^\top P$$

similarly, we have

$$\begin{aligned}\tilde{K}_k &= -(\tilde{R} + \tilde{B}^\top \tilde{P}_{k+1} \tilde{B})^{-1} \tilde{B}^\top \tilde{P}_{k+1} \tilde{A} \\ &= -(hR + h^2 B^\top \tilde{P}_{k+1} B)^{-1} h B^\top \tilde{P}_{k+1} (I + hA) \\ &\rightarrow -R^{-1} B^\top P_{kh}\end{aligned}$$

as $h \rightarrow 0$

Derivation using Lagrange multipliers

pose as constrained problem:

$$\begin{aligned} \text{minimize} \quad & J = \frac{1}{2} \int_0^T \mathbf{x}(\tau)^\top Q \mathbf{x}(\tau) + \mathbf{u}(\tau)^\top R \mathbf{u}(\tau) \, d\tau + \frac{1}{2} \mathbf{x}(T)^\top Q_f \mathbf{x}(T) \\ \text{subject to} \quad & \dot{\mathbf{x}}(t) = A \mathbf{x}(t) + B \mathbf{u}(t), \quad t \in [0, T] \end{aligned}$$

- ▶ optimization variable is *function* $\mathbf{u} : [0, T] \rightarrow \mathbb{R}^m$
- ▶ infinite number of equality constraints, one for each $t \in [0, T]$

introduce Lagrange multiplier *function* $\lambda : [0, T] \rightarrow \mathbb{R}^n$ and form

$$L = J + \int_0^T \lambda(\tau)^\top (A \mathbf{x}(\tau) + B \mathbf{u}(\tau) - \dot{\mathbf{x}}(\tau)) \, d\tau$$

Optimality conditions

from $\nabla_{u(t)} L = Ru(t) + B^T \lambda(t) = 0$ we get $u(t) = -R^{-1} B^T \lambda(t)$

to find $\nabla_{x(t)} L$, we use

$$\int_0^T \lambda(\tau)^T \dot{x}(\tau) d\tau = \lambda(T)^T x(T) - \lambda(0)^T x(0) - \int_0^T \dot{\lambda}(\tau)^T x(\tau) d\tau$$

from $\nabla_{x(t)} L = Qx(t) + A^T \lambda(t) + \dot{\lambda}(t) = 0$ we get

$$\dot{\lambda}(t) = -A^T \lambda(t) - Qx(t)$$

from $\nabla_{x(T)} L = Q_f x(T) - \lambda(T) = 0$, we get $\lambda(T) = Q_f x(T)$

Co-state equations

optimality conditions are

$$\dot{x} = Ax + Bu, \quad x(0) = x_0, \quad \dot{\lambda} = -A^T \lambda - Qx, \quad \lambda(T) = Q_f x(T)$$

using $u(t) = -R^{-1}B^T \lambda(t)$, can write as

$$\frac{d}{dt} \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix}$$

- ▶ $2n \times 2n$ matrix above is called *Hamiltonian* for problem
- ▶ with conditions $x(0) = x_0, \lambda(T) = Q_f x(T)$, called *two-point boundary value problem*

as in discrete-time case, we can show that $\lambda(t) = P_t x(t)$, where

$$-\dot{P}_t = A^T P_t + P_t A - P_t B R^{-1} B^T P_t + Q, \quad P_T = Q_f$$

in other words, value function P_t gives simple relation between x and λ

to show this, we show that $\lambda = Px$ satisfies co-state equation $\dot{\lambda} = -A^T \lambda - Qx$

$$\begin{aligned} \dot{\lambda} &= \frac{d}{dt}(Px) = \dot{P}x + P\dot{x} \\ &= -(Q + A^T P + PA - PBR^{-1}B^T P)x + P(Ax - BR^{-1}B^T \lambda) \\ &= -Qx - A^T Px + PBR^{-1}B^T Px - PBR^{-1}B^T Px \\ &= -Qx - A^T \lambda \end{aligned}$$

Solving Riccati differential equation via Hamiltonian

the (quadratic) Riccati differential equation

$$-\dot{P} = A^T P + P A - P B R^{-1} B^T P + Q$$

and the (linear) Hamiltonian differential equation

$$\frac{d}{dt} \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} = \begin{bmatrix} A & -B R^{-1} B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix}$$

are closely related

$\lambda(t) = P_t x(t)$ suggests that P should have the form $P_t = \lambda(t) x(t)^{-1}$
(but this doesn't make sense unless x and λ are scalars)

consider the Hamiltonian *matrix* (linear) differential equation

$$\frac{d}{dt} \begin{bmatrix} X(t) \\ Y(t) \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} X(t) \\ Y(t) \end{bmatrix}$$

where $X(t), Y(t) \in \mathbb{R}^{n \times n}$

then, $Z(t) = Y(t)X(t)^{-1}$ satisfies Riccati differential equation

$$-\dot{Z} = A^T Z + ZA - ZBR^{-1}B^T Z + Q$$

hence we can solve Riccati DE by solving (linear) matrix Hamiltonian DE, with final conditions $X(T) = I$, $Y(T) = Q_f$, and forming $P(t) = Y(t)X(t)^{-1}$

$$\begin{aligned}
\dot{Z} &= \frac{d}{dt} Y X^{-1} \\
&= \dot{Y} X^{-1} - Y X^{-1} \dot{X} X^{-1} \\
&= (-QX - A^T Y) X^{-1} - Y X^{-1} (AX - BR^{-1} B^T Y) X^{-1} \\
&= -Q - A^T Z - ZA + ZBR^{-1} B^T Z
\end{aligned}$$

where we use two identities:

- ▶ $\frac{d}{dt} (F(t)G(t)) = \dot{F}(t)G(t) + F(t)\dot{G}(t)$
- ▶ $\frac{d}{dt} (F(t)^{-1}) = -F(t)^{-1} \dot{F}(t) F(t)^{-1}$

Infinite horizon LQR

we now consider the infinite horizon cost function

$$J = \int_0^{\infty} x(\tau)^{\top} Q x(\tau) + u(\tau)^{\top} R u(\tau) d\tau$$

we define the value function as

$$V(z) = \min_u \int_0^{\infty} x(\tau)^{\top} Q x(\tau) + u(\tau)^{\top} R u(\tau) d\tau$$

subject to $x(0) = z$, $\dot{x} = Ax + Bu$

we assume that (A, B) is controllable, so V is finite for all z

can show that V is quadratic: $V(z) = z^{\top} P z$, where $P = P^{\top} \geq 0$

optimal u is $u(t) = Kx(t)$, where $K = -R^{-1}B^T P$
(*i.e.*, a constant linear state feedback)

HJ equation is ARE

$$Q + A^T P + PA - PBR^{-1}B^T P = 0$$

which together with $P \geq 0$ characterizes P

can solve as limiting value of Riccati DE, or via direct method

Closed-loop system

with K LQR optimal state feedback gain, closed-loop system is

$$\dot{x} = Ax + Bu = (A + BK)x$$

fact: closed-loop system is stable when (Q, A) observable and (A, B) controllable

we denote eigenvalues of $A + BK$, called *closed-loop eigenvalues*, as $\lambda_1, \dots, \lambda_n$

with assumptions above, $\Re \lambda_i < 0$

Solving ARE via Hamiltonian

$$\begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} I \\ P \end{bmatrix} = \begin{bmatrix} A - BR^{-1}B^T P \\ -Q - A^T P \end{bmatrix} = \begin{bmatrix} A + BK \\ -Q - A^T P \end{bmatrix}$$

and so

$$\begin{bmatrix} I & 0 \\ -P & I \end{bmatrix} \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} I & 0 \\ P & I \end{bmatrix} = \begin{bmatrix} A + BK & -BR^{-1}B^T \\ 0 & -(A + BK)^T \end{bmatrix}$$

where 0 in lower left corner comes from ARE

note that

$$\begin{bmatrix} I & 0 \\ P & I \end{bmatrix}^{-1} = \begin{bmatrix} I & 0 \\ -P & I \end{bmatrix}$$

we see that:

- ▶ eigenvalues of Hamiltonian H are $\lambda_1, \dots, \lambda_n$ and $-\lambda_1, \dots, -\lambda_n$
- ▶ hence, closed-loop eigenvalues are the eigenvalues of H with negative real part

let's assume $A + BK$ is diagonalizable, *i.e.*,

$$T^{-1}(A + BK)T = \Lambda = \mathbf{diag}(\lambda_1, \dots, \lambda_n)$$

then we have $T^\top(-A - BK)^\top T^{-T} = -\Lambda$, so

$$\begin{aligned} & \begin{bmatrix} T^{-1} & 0 \\ 0 & T^\top \end{bmatrix} \begin{bmatrix} A + BK & -BR^{-1}B^\top \\ 0 & -(A + BK)^\top \end{bmatrix} \begin{bmatrix} T & 0 \\ 0 & T^{-T} \end{bmatrix} \\ &= \begin{bmatrix} \Lambda & -T^{-1}BR^{-1}B^\top T^{-T} \\ 0 & -\Lambda \end{bmatrix} \end{aligned}$$

putting it together we get

$$\begin{aligned} & \begin{bmatrix} T^{-1} & 0 \\ 0 & T^\top \end{bmatrix} \begin{bmatrix} I & 0 \\ -P & I \end{bmatrix} H \begin{bmatrix} I & 0 \\ P & I \end{bmatrix} \begin{bmatrix} T & 0 \\ 0 & T^{-T} \end{bmatrix} \\ &= \begin{bmatrix} T^{-1} & 0 \\ -T^\top P & T^\top \end{bmatrix} H \begin{bmatrix} T & 0 \\ PT & T^{-T} \end{bmatrix} \\ &= \begin{bmatrix} \Lambda & -T^{-1}BR^{-1}B^\top T^{-T} \\ 0 & -\Lambda \end{bmatrix} \end{aligned}$$

and so

$$H \begin{bmatrix} T \\ PT \end{bmatrix} = \begin{bmatrix} T \\ PT \end{bmatrix} \Lambda$$

thus, the n columns of $\begin{bmatrix} T \\ PT \end{bmatrix}$ are the eigenvectors of H associated with the stable eigenvalues $\lambda_1, \dots, \lambda_n$

Solving ARE via Hamiltonian

- ▶ find eigenvalues of H , and let $\lambda_1, \dots, \lambda_n$ denote the n stable ones (there are exactly n stable and n unstable ones)
- ▶ find associated eigenvectors v_1, \dots, v_n , and partition as

$$\begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix} \in \mathbb{R}^{2n \times n}$$

- ▶ $P = YX^{-1}$ is unique PSD solution of the ARE

(this is very close to the method used in practice, which does not require $A + BK$ to be diagonalizable)